

GENERALIZED JACOBI FUNCTIONS AND THEIR APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider spectral approximation of fractional differential equations (FDEs). A main ingredient of our approach is to define a new class of generalized Jacobi functions (GJFs), which is intrinsically related to fractional calculus, and can serve as natural basis functions for properly designed spectral methods for FDEs. We establish spectral approximation results for these GJFs in weighted Sobolev spaces involving fractional derivatives. We construct efficient GJF-Petrov-Galerkin methods for a class of prototypical fractional initial value problems (FIVPs) and fractional boundary value problems (FBVPs) of general order, and show that with an appropriate choice of the parameters in GJFs, the resulted linear systems can be sparse and well-conditioned. Moreover, we derive error estimates with convergence rate only depending on the smoothness of data, so truly spectral accuracy can be attained if the data are smooth enough. The idea and results presented in this paper will be useful to deal with more general FDEs associated with Riemann-Liouville or Caputo fractional derivatives.

1. INTRODUCTION

Fractional differential equations appear in the investigation of transport dynamics in complex systems which are governed by the anomalous diffusion and non-exponential relaxation patterns. Related equations of importance are the space/time fractional diffusion equations, the fractional advection-diffusion equations for anomalous diffusion with sources and sinks, the fractional Fokker-Planck equations for anomalous diffusion in an external field, and among others. Progress in the last two decades has demonstrated that many phenomena in various fields of science, mathematics, engineering, bioengineering, and economics are more accurately described by involving fractional derivatives. Nowadays, FDEs are emerging as a new powerful tool for modeling many difficult type of complex systems, i.e., systems with overlapping microscopic and macroscopic scales or systems with long-range time memory and long-range spatial interactions (see, e.g., [24, 23, 13, 6, 7] and the references therein).

There has been a growing interest in the last decades in developing numerical methods for solving FDEs, and a large volume of literature is available on this subject. Generally speaking, two main difficulties for dealing with FDEs are

- (i) fractional derivatives are non-local operators;
- (ii) fractional derivatives involve singular kernel/weight functions, and the solutions of FDEs are usually singular near the boundaries.

Most of the existing numerical methods for FDEs are based on finite difference/finite element methods (cf. [22, 20, 26, 21, 8, 9, 28, 12, 32] and the references therein) which lack the capability

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to effectively deal with the aforementioned difficulties, as they are based on “local” operations, and are not well-suited for problems with singular kernels/weights. In particular, due to the non-local nature of the fractional derivatives, they all lead to full and dense matrices which are expensive to calculate and to invert. Recently, some interesting ideas have been proposed to overcome these difficulties. For instance, Wang and Basu [29] proposed a fast finite-difference method by carefully analyzing the structure of the coefficient matrices of the resulted linear systems, and delicately decomposing them into a combination of sparse and structured dense matrices.

There exist also limited but very promising efforts in developing spectral methods for solving FDEs (see, e.g., [18, 19, 17, 30, 31]). The spectral method appears to be a natural approach, since it is global, which should be better suited for non-local problems. Most notably, Zayernouri and Karniadakis [30] proposed to use polyfractomials as basis functions, which are eigenfunctions of a fractional Sturm-Liouville operator, and result in sparse matrices for some simple model equations. Preliminary results in [30] showed that this new approach could lead to several orders of magnitude saving in CPU and memory for some model FDEs. However, there is no error analysis available for the approximation properties of polyfractomials, and the algorithms therein do not necessarily lead to spectral convergence for problems with smooth data but non-smooth solution which is typical for FDEs.

The second difficulty is largely ignored in the literature. Typically, the solution and data of a FDE are not in the same type of Sobolev spaces, which is in distinctive contrast with usual DEs. Consequently, they should be approximated by different tools, and the error estimates should be measured in norms of different types of spaces. Indeed, given smooth data, the solution of a FDE only has limited regularity in the usual Sobolev spaces. However, existing error estimates for FDEs, either finite differences, finite elements or spectral methods, are all based on the usual approach, namely, the errors are performed in the framework of usual Sobolev spaces. Hence, it is not surprising to see that most existing methods and the related error estimates only lead to poor convergence rate for typical FDEs, unless one manufactures a smooth exact solution, directly uses a polynomial-based method, and then carefully deals with the singular data.

The purpose of this paper is to develop and analyze efficient spectral methods which can effectively address the above two issues for a class of prototypical FDEs. The main strategies and contributions are highlighted as follows.

- We introduce a new class of GJFs with two parameters, which can be tuned to match singularity of the underlying solution, and simultaneously produce sparse linear systems. More importantly, such GJFs enjoy attractive fractional calculus properties and remarkable approximability to functions with singular behaviour at boundaries.
- We derive optimal approximation results for these GJFs in suitably weighted spaces involving fractional derivatives, and obtain error estimates for the proposed GJF-Petrov-Galerkin approaches with convergence rate only depending on smoothness of the data (characterised by usual Sobolev norms). Thus, truly spectral accuracy can be achieved for some model FDEs with sufficient smooth data.
- We point out that the GJFs, including generalised Jacobi polynomials (GJPs) as special cases, have been first introduced in [10, 11] for solutions of usual BVPs. Here, we modify the original definition, especially the range of the parameters, which opens up new applications in solving FDEs. We also remark that GJFs with parameters in $(0, 1)$ have direct bearing on the Jacobi polyfractomials in [30]. The major difference from these relevant existing ones lies in that the new GJFs are built upon Jacobi polynomials *with real parameters*. This is essential for both algorithm development and error analysis.

While we shall only consider some prototypical FIVPs and FBVPs of general order, we position this work as the first but important step towards developing efficient spectral methods for more complicated FDEs involving Riemann-Liouville or Caputo fractional derivatives.

The paper is organized as follows. In the next section, we make necessary preparations by recalling basic properties of Jacobi polynomials with real parameters, and introducing the important

Bateman fractional integral formula. In Section 3, we define the GJFs and derive their essential properties, particularly, including fractional calculus properties. In Section 4, we establish the approximation results for these GJFs. In Section 5, we construct efficient GJF-Petrov-Galerkin methods for a class of prototypical FDEs, conduct error analysis and present ample supporting numerical results. In the final section, we extend the most important Riemann-Liouville fractional derivative formulas to the Caputo fractional derivatives, and conclude the paper with a few remarks.

2. PRELIMINARIES

In this section, we review basics of fractional integrals/derivatives, and recall relevant properties of the Jacobi polynomials with real parameters. In particular, we introduce the Bateman fractional integral formula, which plays a very important role in the forthcoming algorithm development and analysis.

2.1. Fractional integrals and derivatives. Let \mathbb{N} and \mathbb{R} be the set of positive integers and real numbers, respectively. Denote

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad \mathbb{R}^+ := \{a \in \mathbb{R} : a > 0\}, \quad \mathbb{R}_0^+ := \{0\} \cup \mathbb{R}^+. \quad (2.1)$$

We first recall the definitions of the fractional integrals and fractional derivatives in the sense of Riemann-Liouville and Caputo (see, e.g., [24, 6]). To fix the idea, we restrict our attentions to the interval $(-1, 1)$. It is clear that all formulas and properties can be formulated on a general interval (a, b) .

Definition 2.1 (Fractional integrals and derivatives). For $\rho \in \mathbb{R}^+$, the left and right fractional integrals are respectively defined as

$$I_-^\rho v(x) = \frac{1}{\Gamma(\rho)} \int_{-1}^x \frac{v(y)}{(x-y)^{1-\rho}} dy, \quad x > -1; \quad I_+^\rho v(x) = \frac{1}{\Gamma(\rho)} \int_x^1 \frac{v(y)}{(y-x)^{1-\rho}} dy, \quad x < 1, \quad (2.2)$$

where $\Gamma(\cdot)$ is the usual Gamma function.

For $s \in [k-1, k)$ with $k \in \mathbb{N}$, the left-sided Riemann-Liouville fractional derivative (LRLFD) of order s is defined by

$$D_-^s v(x) = \frac{1}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_{-1}^x \frac{v(y)}{(x-y)^{s-k+1}} dy, \quad x \in \Lambda := (-1, 1), \quad (2.3)$$

and the right-sided Riemann-Liouville fractional derivative (RRLFD) of order s is defined by

$$D_+^s v(x) = \frac{(-1)^k}{\Gamma(k-s)} \frac{d^k}{dx^k} \int_x^1 \frac{v(y)}{(y-x)^{s-k+1}} dy, \quad x \in \Lambda. \quad (2.4)$$

For $s \in [k-1, k)$ with $k \in \mathbb{N}$, the left-sided Caputo fractional derivatives (LCFD) of order s is defined by

$${}^C D_-^s v(x) := \frac{1}{\Gamma(k-s)} \int_{-1}^x \frac{v^{(k)}(y)}{(x-y)^{s-k+1}} dy, \quad x \in \Lambda, \quad (2.5)$$

and the right-sided Caputo fractional derivatives (RCFD) of order s is defined by

$${}^C D_+^s v(x) := \frac{(-1)^k}{\Gamma(k-s)} \int_x^1 \frac{v^{(k)}(y)}{(y-x)^{s-k+1}} dy, \quad x \in \Lambda. \quad (2.6)$$

It is clear that for any $k \in \mathbb{N}_0$,

$$D_-^k = D^k, \quad D_+^k = (-1)^k D^k, \quad \text{where } D^k := d^k/dx^k. \quad (2.7)$$

Thus, we can define the RLFD as

$$D_-^s v(x) = D^k I_-^{k-s} v(x), \quad D_+^s v(x) = (-1)^k D^k I_+^{k-s} v(x). \quad (2.8)$$

According to [6, Thm. 2.14], we have that for any absolutely integrable function v , and real $s \geq 0$,

$$D_{\pm}^s I_{\pm}^s v(x) = v(x), \quad \text{a.e. in } \Lambda. \quad (2.9)$$

The following lemma shows the relationship between the Riemann-Liouville and Caputo fractional derivatives (see, e.g., [24, Ch. 2]).

Lemma 2.1. *For $s \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$$D_-^s v(x) = {}^C D_-^s v(x) + \sum_{j=0}^{k-1} \frac{v^{(j)}(-1)}{\Gamma(1+j-s)} (1+x)^{j-s}; \quad (2.10a)$$

$$D_+^s v(x) = {}^C D_+^s v(x) + \sum_{j=0}^{k-1} \frac{(-1)^j v^{(j)}(1)}{\Gamma(1+j-s)} (1-x)^{j-s}. \quad (2.10b)$$

Remark 2.1. In the above, the Gamma function with negative, non-integer argument should be understood by the Euler reflection formula (cf. [1]):

$$\Gamma(1+j-s) = \frac{\pi}{\sin(\pi(1+j-s))} \frac{1}{\Gamma(s-j)}, \quad s \in (k-1, k), \quad 1 \leq j \leq k-2.$$

Note that if $s = k-1$, then $\Gamma(1+j-s) = \infty$ for all $0 \leq j \leq k-2$, so the summations in the above reduce to $v^{(k-1)}(\pm 1)$, respectively. \square

Remark 2.2. We observe immediately from (2.10) that for $s \in [k-1, k)$ with $k \in \mathbb{N}$,

$$D_{\pm}^s v(x) = {}^C D_{\pm}^s v(x), \quad \text{if } v^{(j)}(\pm 1) = 0, \quad 0 \leq j \leq k-1. \quad (2.11)$$

The rule of fractional integration by parts (see, e.g., [14]) will also be used subsequently.

Lemma 2.2. *For $s \in [k-1, k)$ with $k \in \mathbb{N}$, we have*

$$(D_-^s u, v) = (u, {}^C D_+^s v) + \sum_{j=0}^{k-1} (-1)^j v^{(j)}(x) D^{k-j-1} I_-^{k-s} u(x) \Big|_{x=-1}^{x=1}; \quad (2.12a)$$

$$(D_+^s u, v) = (u, {}^C D_-^s v) + \sum_{j=0}^{k-1} (-1)^{k-j} v^{(j)}(x) D^{k-j-1} I_+^{k-s} u(x) \Big|_{x=-1}^{x=1}, \quad (2.12b)$$

where (\cdot, \cdot) is the L^2 -inner product.

2.2. Jacobi polynomials with real parameters. Much of our discussion later will make use of Jacobi polynomials with real parameters. Below, we review their relevant properties.

Recall the hypergeometric function (cf. [1]):

$${}_2F_1(a, b; c; x) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{x^j}{j!}, \quad |x| < 1, \quad a, b, c \in \mathbb{R}, \quad -c \notin \mathbb{N}_0, \quad (2.13)$$

where the rising factorial in the Pochhammer symbol, for $a \in \mathbb{R}$ and $j \in \mathbb{N}_0$, is defined by:

$$(a)_0 = 1; \quad (a)_j := a(a+1) \cdots (a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}, \quad \text{for } j \geq 1. \quad (2.14)$$

If a or b is a negative integer, then it reduces to a polynomial.

The classical Jacobi polynomials are defined for parameters $\alpha, \beta > -1$. The Jacobi polynomials can also be defined for $\alpha \leq -1$ and/or $\beta \leq -1$ as in Szegő [27, (4.21.2)]:

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(\alpha+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \\ &= \frac{(\alpha+1)_n}{n!} + \sum_{j=1}^{n-1} \frac{(n+\alpha+\beta+1)_j (\alpha+j+1) \cdots (\alpha+n)}{j!(n-j)!} \left(\frac{x-1}{2}\right)^j \\ &\quad + \frac{(n+\alpha+\beta+1)_n}{n!} \left(\frac{x-1}{2}\right)^n, \quad n \geq 1, \end{aligned} \quad (2.15)$$

and $P_0^{(\alpha, \beta)}(x) \equiv 1$. Note that $P_n^{(\alpha, \beta)}(x)$ is always a polynomial in x for all $\alpha, \beta \in \mathbb{R}$.

Many properties of the classical Jacobi polynomial (with $\alpha, \beta > -1$) can be extended to the general case (with $\alpha, \beta \in \mathbb{R}$), see [27, P. 62-67]. In particular, there hold

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x); \quad P_n^{(\alpha, \beta)}(1) = \frac{(\alpha+1)_n}{n!}. \quad (2.16)$$

Thus, we have the alternative representation:

$$P_n^{(\alpha, \beta)}(x) = (-1)^n \frac{(\beta+1)_n}{n!} {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right), \quad n \geq 1. \quad (2.17)$$

Since the leading coefficient of $P_n^{(\alpha, \beta)}(x)$ is $(n+\alpha+\beta+1)_n / (2^n n!)$ (see (2.15)), its degree is less than n , when $n+\alpha+\beta \in \{-1, \dots, -n\}$ (i.e., $(n+\alpha+\beta+1)_n = 0$). We also refer to [27, (4.22.3)] for details of the reduction. Throughout this paper, we assume that

$$-(n+\alpha+\beta) \notin \mathbb{N}, \quad \forall n \geq 1, \quad (2.18)$$

so $P_n^{(\alpha, \beta)}(x)$ is always a polynomial of degree n . Under the condition (2.18), the Jacobi polynomials defined by (2.15) can be computed by the same three-term recurrence relation as the classical Jacobi polynomials:

$$\begin{aligned} P_{n+1}^{(\alpha, \beta)}(x) &= (a_n^{\alpha, \beta} x - b_n^{\alpha, \beta}) P_n^{(\alpha, \beta)}(x) - c_n^{\alpha, \beta} P_{n-1}^{(\alpha, \beta)}(x), \quad n \geq 1, \\ P_0^{(\alpha, \beta)}(x) &= 1, \quad P_1^{(\alpha, \beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \end{aligned} \quad (2.19)$$

where

$$a_n^{\alpha, \beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}, \quad (2.20a)$$

$$b_n^{\alpha, \beta} = \frac{(\beta^2 - \alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \quad (2.20b)$$

$$c_n^{\alpha, \beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \quad (2.20c)$$

We particularly look at the Jacobi polynomials with one or both parameters being negative integers. If $\alpha = -l$ (with $l \in \mathbb{N}$), $\beta \in \mathbb{R}$ and $n \geq l \geq 1$, we have that (see [27, (4.22.2)])

$$P_n^{(-l, \beta)}(x) = d_n^{l, \beta} \left(\frac{x-1}{2}\right)^l P_{n-l}^{(l, \beta)}(x), \quad \text{where} \quad d_n^{l, \beta} = \frac{(n-l)!(\beta+n-l+1)_l}{n!}. \quad (2.21)$$

Similarly, for $\beta = -m$, we find from (2.16) and (2.21) that

$$P_n^{(\alpha, -m)}(x) = d_n^{m, \alpha} \left(\frac{x+1}{2}\right)^m P_{n-m}^{(\alpha, m)}(x), \quad n \geq m \geq 1, \quad \alpha \in \mathbb{R}. \quad (2.22)$$

Therefore, we deduce from (2.21)-(2.22) that for $n \geq l+m$ and $l, m \in \mathbb{N}$,

$$P_n^{(-l, -m)}(x) = \left(\frac{x-1}{2}\right)^l \left(\frac{x+1}{2}\right)^m P_{n-l-m}^{(l, m)}(x), \quad (2.23)$$

where we used the fact $d_n^{l, -m} d_{n-l}^{m, l} = 1$.

For $\alpha, \beta > -1$, the (classical) Jacobi polynomials are orthogonal with respect to the Jacobi weight function: $\omega^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta$, namely,

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_{n'}^{(\alpha, \beta)}(x) \omega^{(\alpha, \beta)}(x) dx = \gamma_n^{(\alpha, \beta)} \delta_{nn'}, \quad (2.24)$$

where $\delta_{nn'}$ is the Dirac Delta symbol, and the normalization constant is given by

$$\gamma_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) n! \Gamma(n+\alpha+\beta+1)}. \quad (2.25)$$

However, the orthogonality does not carry over to the general case. We refer to [16] and [15, Ch. 3] for details.

2.3. Bateman fractional integral formula. We recall the fractional integral formula of hypergeometric functions due to Bateman [5] (also see [3, P. 313]): for real $c, \rho \geq 0$,

$${}_2F_1(a, b; c + \rho; x) = \frac{\Gamma(c + \rho)}{\Gamma(c) \Gamma(\rho)} x^{1-(c+\rho)} \int_0^x t^{c-1} (x-t)^{\rho-1} {}_2F_1(a, b; c; t) dt, \quad |x| < 1, \quad (2.26)$$

where the hypergeometric function ${}_2F_1$ is defined in (2.13).

The following formulas, derived from (2.15) and (2.26) (cf. [27, P. 96]), are indispensable for the subsequent discussion.

Lemma 2.3. *Let $\rho \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

(i) *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$$(1-x)^{\alpha+\rho} \frac{P_n^{(\alpha+\rho, \beta-\rho)}(x)}{P_n^{(\alpha+\rho, \beta-\rho)}(1)} = \frac{\Gamma(\alpha+\rho+1)}{\Gamma(\alpha+1) \Gamma(\rho)} \int_x^1 \frac{(1-y)^\alpha}{(y-x)^{1-\rho}} \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)} dy. \quad (2.27)$$

(ii) *For $\alpha \in \mathbb{R}$ and $\beta > -1$,*

$$(1+x)^{\beta+\rho} \frac{P_n^{(\alpha-\rho, \beta+\rho)}(x)}{P_n^{(\beta+\rho, \alpha-\rho)}(1)} = \frac{\Gamma(\beta+\rho+1)}{\Gamma(\beta+1) \Gamma(\rho)} \int_{-1}^x \frac{(1+y)^\beta}{(x-y)^{1-\rho}} \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\beta, \alpha)}(1)} dy. \quad (2.28)$$

Remark 2.3. The formulas (2.27)-(2.28) can be found in several classical books on orthogonal polynomials, but it appears that their derivation is not well described. In fact, taking $a = -n, b = n + \alpha + \beta + 1, c = \alpha + 1$ and $t = (1-y)/2$ in (2.26), we obtain the formula (2.27) from (2.15). Similarly, (2.28) follows from (2.16) and (2.27). \square

Using the notation in Definition 2.1 and working out the constants by (2.16), we can rewrite the formulas in Lemma 2.3 as follows.

Lemma 2.4. *Let $\rho \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

• *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$$I_+^\rho \{ (1-x)^\alpha P_n^{(\alpha, \beta)}(x) \} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha+\rho+1)} (1-x)^{\alpha+\rho} P_n^{(\alpha+\rho, \beta-\rho)}(x). \quad (2.29)$$

• *For $\alpha \in \mathbb{R}$ and $\beta > -1$,*

$$I_-^\rho \{ (1+x)^\beta P_n^{(\alpha, \beta)}(x) \} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\rho+1)} (1+x)^{\beta+\rho} P_n^{(\alpha-\rho, \beta+\rho)}(x). \quad (2.30)$$

Thanks to (2.9), we obtain from Lemma 2.4 the following useful “inverse” rules.

Lemma 2.5. *Let $s \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

• *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$$D_+^s \{ (1-x)^{\alpha+s} P_n^{(\alpha+s, \beta-s)}(x) \} = \frac{\Gamma(n+\alpha+s+1)}{\Gamma(n+\alpha+1)} (1-x)^\alpha P_n^{(\alpha, \beta)}(x). \quad (2.31)$$

- For $\alpha \in \mathbb{R}$ and $\beta > -1$,

$$D_-^s \{(1+x)^{\beta+s} P_n^{(\alpha-s, \beta+s)}(x)\} = \frac{\Gamma(n+\beta+s+1)}{\Gamma(n+\beta+1)} (1+x)^\beta P_n^{(\alpha, \beta)}(x). \quad (2.32)$$

Observe that if $\alpha = 0$ in (2.31), the fractional derivative operator D_+^s takes $(1-x)^s P_n^{(s, \beta-s)}(x)$ to the polynomial $P_n^{(0, \beta)}(x)$. Conversely, if $\alpha+s = k \in \mathbb{N}_0$, D_+^s takes the polynomial $(1-x)^k P_n^{(k, \beta-s)}(x)$ to $(1-x)^{s-k} P_n^{(s-k, \beta)}(x)$. Such remarkable properties are essential for efficient spectral algorithms to be developed later. We next show that the above non-polynomial functions are intimately related to the generalized Jacobi functions introduced in [11]. Moreover, the Jacobi poly-fractionomials first introduced in [30] also have direct bearing on these basis functions when $s \in (0, 1)$.

3. GENERALIZED JACOBI FUNCTIONS

In this section, we modify the definition of two subclasses of GJFs in [11], leading to the basis functions of interest, which will be still dubbed as GJFs. We shall demonstrate in Section 5 that spectral algorithms using GJF as basis functions produce spectral accurate solutions for a class of prototypical fractional differential equations.

3.1. Definition of GJFs.

Definition 3.1 (Generalized Jacobi functions). *Define*

$$+J_n^{(-\alpha, \beta)}(x) := (1-x)^\alpha P_n^{(\alpha, \beta)}(x), \quad \text{for } \alpha > -1, \beta \in \mathbb{R}, \quad (3.1)$$

and

$$-J_n^{(\alpha, -\beta)}(x) := (1+x)^\beta P_n^{(\alpha, \beta)}(x), \quad \text{for } \alpha \in \mathbb{R}, \beta > -1, \quad (3.2)$$

for all $x \in \Lambda$ and $n \geq 0$.

Remark 3.1. Note that the above definitions modified the classical Jacobi polynomials in the range of $-1 < \alpha, \beta < 1$. \square

Recall the GJFs introduced in [11, (2.7)]:

$$j_n^{(\alpha, \beta)}(x) = \begin{cases} (1-x)^{-\alpha} (1+x)^{-\beta} P_{\hat{n}}^{(-\alpha, -\beta)}(x), & (\alpha, \beta) \in \aleph_1, \quad \hat{n} = n - [-\alpha] - [-\beta], \\ (1-x)^{-\alpha} P_{\hat{n}}^{(-\alpha, \beta)}(x), & (\alpha, \beta) \in \aleph_2, \quad \hat{n} = n - [-\alpha], \\ (1+x)^{-\beta} P_{\hat{n}}^{(\alpha, -\beta)}(x), & (\alpha, \beta) \in \aleph_3, \quad \hat{n} = n - [-\beta], \\ P_n^{(\alpha, \beta)}(x), & (\alpha, \beta) \in \aleph_4, \end{cases} \quad (3.3)$$

where

$$\begin{aligned} \aleph_1 &= \{(\alpha, \beta) : \alpha, \beta \leq -1\}, & \aleph_2 &= \{(\alpha, \beta) : \alpha \leq -1, \beta > -1\}, \\ \aleph_3 &= \{(\alpha, \beta) : \alpha > -1, \beta \leq -1\}, & \aleph_4 &= \{(\alpha, \beta) : \alpha, \beta > -1\}. \end{aligned}$$

We elaborate below on the connection and difference between the new GJFs and the GJFs defined in (3.3).

- Comparing (3.1)-(3.2) with (3.3), we find

$$\begin{aligned} +J_n^{(-\alpha, \beta)}(x) &= j_{n+[\alpha]}^{(-\alpha, \beta)}(x), \quad \text{if } \alpha \geq 1, \beta > -1; \\ -J_n^{(\alpha, -\beta)}(x) &= j_{n+[\beta]}^{(\alpha, -\beta)}(x), \quad \text{if } \alpha > -1, \beta \geq 1. \end{aligned} \quad (3.4)$$

- By (2.21)-(2.22), we find from (3.1)-(3.2) that for any $\alpha > -1, k \in \mathbb{N}_0$ and $n \geq k$,

$$\begin{aligned} +J_n^{(-\alpha, -k)}(x) &= 2^{-k} d_n^{k, \alpha} (1-x)^\alpha (1+x)^k P_{n-k}^{(\alpha, k)}(x); \\ -J_n^{(k, -\alpha)}(x) &= (-1)^k 2^{-k} d_n^{k, \alpha} (1-x)^k (1+x)^\alpha P_{n-k}^{(k, \alpha)}(x), \end{aligned} \quad (3.5)$$

which, compared with (3.3), implies that for $\alpha \geq 1$ and $n \geq k \geq 1$,

$$+J_n^{(-\alpha, -k)}(x) = 2^{-k} d_n^{k, \alpha} j_{n+[\alpha]}^{(-\alpha, -k)}(x); \quad -J_n^{(-\alpha, -k)}(x) = (-1)^k 2^{-k} d_n^{k, \alpha} j_{n+[\alpha]}^{(-k, -\alpha)}(x). \quad (3.6)$$

Here, the constant $d_n^{k, \alpha}$ is defined in (2.21).

We see that we modified the definition of GJFs in [11] for the parameters in the ranges other than those specified in (3.4) and (3.6). Indeed, this opens up new applicability of the GJFs in solving fractional differential equations, see Section 5.

3.2. Properties of GJFs. One verifies readily from (2.16) and Definition 3.1 that for $\alpha > -1$ and $\beta \in \mathbb{R}$,

$$+J_n^{(-\alpha, \beta)}(-x) = (-1)^n -J_n^{(\beta, -\alpha)}(x), \quad (3.7)$$

and for $-1 < \alpha < 1$, there holds the reflection property:

$$+J_n^{(-\alpha, -\alpha)}(x) = (1 - x^2)^\alpha -J_n^{(\alpha, \alpha)}(x). \quad (3.8)$$

If $-(n + \alpha + \beta) \notin \mathbb{N}$, we can use (2.19) to evaluate $+J_n^{(-\alpha, \beta)}$ recursively:

$$\begin{aligned} +J_{n+1}^{(-\alpha, \beta)}(x) &= (a_n^{\alpha, \beta} x - b_n^{\alpha, \beta}) +J_n^{(-\alpha, \beta)}(x) - c_n^{\alpha, \beta} +J_{n-1}^{(-\alpha, \beta)}(x), \quad n \geq 1, \\ +J_0^{(-\alpha, \beta)}(x) &= (1 - x)^\alpha, \quad +J_1^{(-\alpha, \beta)}(x) = ((\alpha + \beta + 2)x + \alpha - \beta)(1 - x)^\alpha / 2, \end{aligned} \quad (3.9)$$

where $a_n^{\alpha, \beta}, b_n^{\alpha, \beta}, c_n^{\alpha, \beta}$ are defined in (2.20). Accordingly, we can compute $-J_n^{(\alpha, -\beta)}(x)$ by (3.7).

We now study the orthogonality of GJFs. It follows straightforwardly from (2.24) and Definition 3.1 that for $\alpha, \beta > -1$,

$$\begin{aligned} &\int_{-1}^1 +J_n^{(-\alpha, \beta)}(x) +J_{n'}^{(-\alpha, \beta)}(x) \omega^{(-\alpha, \beta)}(x) dx \\ &= \int_{-1}^1 -J_n^{(\alpha, -\beta)}(x) -J_{n'}^{(\alpha, -\beta)}(x) \omega^{(\alpha, -\beta)}(x) dx = \gamma_n^{(\alpha, \beta)} \delta_{nn'}, \end{aligned} \quad (3.10)$$

where $\gamma_n^{(\alpha, \beta)}$ is defined in (2.25). Similarly, by (2.24) and (3.5), we have that for $\alpha > -1$ and $k \in \mathbb{N}$,

$$\begin{aligned} &\int_{-1}^1 +J_n^{(-\alpha, -k)}(x) +J_{n'}^{(-\alpha, -k)}(x) \omega^{(-\alpha, -k)}(x) dx \\ &= \int_{-1}^1 -J_n^{(-k, -\alpha)}(x) -J_{n'}^{(-k, -\alpha)}(x) \omega^{(-k, -\alpha)}(x) dx = \gamma_n^{(\alpha, -k)} \delta_{nn'}, \quad n, n' \geq k, \end{aligned} \quad (3.11)$$

where we used the fact

$$\gamma_n^{(\alpha, -k)} = 2^{-2k} (d_n^{k, \alpha})^2 \gamma_{n-k}^{(\alpha, k)}.$$

Next, we discuss the fractional calculus properties of GJFs. The following fractional derivative formulas can be derived straightforwardly from Lemma 2.5 and Definition 3.1.

Theorem 3.1. *Let $s \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

- For $\alpha > s - 1$ and $\beta \in \mathbb{R}$,

$$D_+^s \{ +J_n^{(-\alpha, \beta)}(x) \} = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha - s + 1)} +J_n^{(-\alpha + s, \beta + s)}(x). \quad (3.12)$$

- For $\alpha \in \mathbb{R}$ and $\beta > s - 1$,

$$D_-^s \{ -J_n^{(\alpha, -\beta)}(x) \} = \frac{\Gamma(n + \beta + 1)}{\Gamma(n + \beta - s + 1)} -J_n^{(\alpha + s, -\beta + s)}(x). \quad (3.13)$$

Some remarks on Theorem 3.1 are in order.

- If $\alpha - s > -1$ and $\beta + s > -1$ with $s \in \mathbb{R}^+$, then by (3.10) and (3.12), $\{D_+^s + J_n^{(-\alpha, \beta)}\}$ are mutually orthogonal with respect to the weight function $\omega^{(-\alpha+s, \beta+s)}(x)$. Similarly, $\{D_-^s - J_n^{(\alpha, -\beta)}\}$ are mutually orthogonal with respect to $\omega^{(\alpha+s, -\beta+s)}(x)$, when $\alpha + s > -1$ and $\beta - s > -1$.
- A very important special case of (3.12) is that for $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$D_+^\alpha \{+J_n^{(-\alpha, \beta)}(x)\} = \frac{\Gamma(n + \alpha + 1)}{n!} + J_n^{(0, \alpha + \beta)}(x) = \frac{\Gamma(n + \alpha + 1)}{n!} P_n^{(0, \alpha + \beta)}(x). \quad (3.14)$$

Similarly, by (3.13), we have that for $\alpha \in \mathbb{R}$ and real $\beta > 0$,

$$D_-^\beta \{-J_n^{(\alpha, -\beta)}(x)\} = \frac{\Gamma(n + \beta + 1)}{n!} P_n^{(\alpha + \beta, 0)}(x). \quad (3.15)$$

These two formulas indicate that performing a suitable order of fractional derivatives on GJFs leads to polynomials.

The analysis of the approximability of GJFs essentially relies on the orthogonality of fractional derivatives of GJFs. To study this, we first recall the derivative formula of the classical Jacobi polynomials (see, e.g., [25, P. 72]): for $\alpha, \beta > -1$ and $n \geq l$,

$$D^l P_n^{(\alpha, \beta)}(x) = \kappa_{n,l}^{(\alpha, \beta)} P_{n-l}^{(\alpha+l, \beta+l)}(x), \quad \text{where } \kappa_{n,l}^{(\alpha, \beta)} := \frac{\Gamma(n + \alpha + \beta + l + 1)}{2^l \Gamma(n + \alpha + \beta + 1)}. \quad (3.16)$$

Noting that $D_\pm^{s+l} = (\mp 1)^l D^l D_\pm^s$, we derive from (2.24) and (3.14)-(3.16) the following orthogonality.

- For $\alpha > 0$ and $\alpha + \beta > -1$,

$$\int_{-1}^1 D_+^{\alpha+l} + J_n^{(-\alpha, \beta)}(x) D_+^{\alpha+l} + J_{n'}^{(-\alpha, \beta)}(x) \omega^{(l, \alpha + \beta + l)}(x) dx = h_{n,l}^{(\alpha, \beta)} \delta_{nn'}, \quad n, n' \geq l \geq 0, \quad (3.17)$$

where

$$\begin{aligned} h_{n,l}^{(\alpha, \beta)} &:= \frac{\Gamma^2(n + \alpha + 1)}{(n!)^2} (\kappa_{n,l}^{(0, \alpha + \beta)})^2 \gamma_{n-l}^{(l, \alpha + \beta + l)} \\ &= \frac{2^{\alpha + \beta + 1} \Gamma^2(n + \alpha + 1) \Gamma(n + \alpha + \beta + l + 1)}{(2n + \alpha + \beta + 1) n! (n-l)! \Gamma(n + \alpha + \beta + 1)}. \end{aligned} \quad (3.18)$$

- For $\alpha + \beta > -1$ and $\beta > 0$,

$$\int_{-1}^1 D_-^{\beta+l} - J_n^{(\alpha, -\beta)}(x) D_-^{\beta+l} - J_{n'}^{(\alpha, -\beta)}(x) \omega^{(\alpha + \beta + l, l)}(x) dx = h_{n,l}^{(\beta, \alpha)} \delta_{nn'}, \quad n, n' \geq l \geq 0. \quad (3.19)$$

Another attractive property of GJFs is that they are eigenfunctions of fractional Sturm-Liouville-type equations. To show this, we define the fractional Sturm-Liouville-type operators:

$${}^+\mathcal{L}_{\alpha, \beta}^{2s} u := \omega^{(\alpha, -\beta)} D_-^s \{\omega^{(-\alpha+s, \beta+s)} D_+^s u\}; \quad {}^-\mathcal{L}_{\alpha, \beta}^{2s} u := \omega^{(-\alpha, \beta)} D_+^s \{\omega^{(\alpha+s, -\beta+s)} D_-^s u\}. \quad (3.20)$$

Theorem 3.2. *Let $s \in \mathbb{R}^+$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.*

- For $\alpha > s - 1$ and $\beta > -1$,

$${}^+\mathcal{L}_{\alpha, \beta}^{2s} + J_n^{(-\alpha, \beta)}(x) = \lambda_{n,s}^{(\alpha, \beta)} + J_n^{(-\alpha, \beta)}(x), \quad (3.21)$$

where

$$\lambda_{n,s}^{(\alpha, \beta)} := \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + \alpha - s + 1)} \frac{\Gamma(n + \beta + s + 1)}{\Gamma(n + \beta + 1)}. \quad (3.22)$$

- For $\alpha > -1$ and $\beta > s - 1$,

$${}^-\mathcal{L}_{\alpha, \beta}^{2s} - J_n^{(\alpha, -\beta)}(x) = \lambda_{n,s}^{(\beta, \alpha)} - J_n^{(\alpha, -\beta)}(x). \quad (3.23)$$

Proof. By Definition 3.1 and (3.12), we have that for $\alpha > s - 1$,

$$(1-x)^{-\alpha+s}(1+x)^{\beta+s}D_+^s\{+J_n^{(-\alpha,\beta)}(x)\} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-s+1)}(1+x)^{\beta+s}P_n^{(\alpha-s,\beta+s)}(x). \quad (3.24)$$

Applying D_-^s on both sides of the above identity and tracking the constants, we derive from (2.32) that for $\beta > -1$,

$$\begin{aligned} D_-^s\{\omega^{(-\alpha+s,\beta+s)}(x)D_+^s\{+J_n^{(-\alpha,\beta)}(x)\}\} &= \lambda_{n,s}^{(\alpha,\beta)}(1+x)^\beta P_n^{(\alpha,\beta)}(x) \\ &= \lambda_{n,s}^{(\alpha,\beta)}\omega^{(-\alpha,\beta)}(x)+J_n^{(-\alpha,\beta)}(x). \end{aligned}$$

This yields (3.21).

The property (3.23) can be proved in a very similar fashion. \square

Remark 3.2. The above results can be viewed as an extension of the standard Sturm-Liouville problems of GJFs to the fractional derivative case. In [11], we showed that GJFs defined therein are the eigenfunctions of the standard Sturm-Liouville problems. \square

Remark 3.3. We derive immediately from (3.22) and the Stirling's formula (see (4.24) below) that for fixed s, α, β ,

$$\lambda_{n,s}^{(\alpha,\beta)} = O(n^{2s}), \quad \text{for } n \gg 1.$$

When $s \rightarrow 1$, this recovers the $O(n^2)$ growth of eigenvalues of the standard Sturm-Liouville problem. \square

Note that the fractional Sturm-Liouville operators defined in (3.20) are not self-adjoint in general. However, the singular fractional Sturm-Liouville problems are self-adjoint, when $s \in (0, 1)$.

Corollary 3.1. Let $s \in (0, 1)$, $n \in \mathbb{N}_0$ and $x \in \Lambda$.

- For $0 < \alpha < s$ and $\beta > -s$, we have that in (3.21),

$$+\mathcal{L}_{\alpha,\beta}^{2s}+J_n^{(-\alpha,\beta)} = \omega^{(\alpha,-\beta)}D_-^s\{\omega^{(-\alpha+s,\beta+s)}CD_+^s+J_n^{(-\alpha,\beta)}\}, \quad (3.25)$$

and

$$\begin{aligned} (+\mathcal{L}_{\alpha,\beta}^{2s}+J_n^{(-\alpha,\beta)}, +J_m^{(-\alpha,\beta)})_{\omega^{(-\alpha,\beta)}} &= (CD_+^s+J_n^{(-\alpha,\beta)}, CD_+^s+J_m^{(-\alpha,\beta)})_{\omega^{(-\alpha-s,\beta+s)}} \\ &= (+J_n^{(-\alpha,\beta)}, +\mathcal{L}_{\alpha,\beta}^{2s}+J_m^{(-\alpha,\beta)})_{\omega^{(-\alpha,\beta)}} = \lambda_{n,s}^{(\alpha,\beta)}\gamma_n^{(-\alpha,\beta)}\delta_{nm}. \end{aligned} \quad (3.26)$$

- Similarly, for $\alpha > -s$ and $0 < \beta < s$, we have that in (3.23),

$$-\mathcal{L}_{\alpha,\beta}^{2s}-J_n^{(\alpha,-\beta)} = \omega^{(-\alpha,\beta)}D_+^s\{\omega^{(\alpha+s,-\beta+s)}CD_-^s-J_n^{(\alpha,-\beta)}\}, \quad (3.27)$$

and

$$\begin{aligned} (-\mathcal{L}_{\alpha,\beta}^{2s}-J_n^{(\alpha,-\beta)}, -J_m^{(\alpha,-\beta)})_{\omega^{(\alpha,-\beta)}} &= (CD_-^s-J_n^{(\alpha,-\beta)}, CD_-^s-J_m^{(\alpha,-\beta)})_{\omega^{(\alpha+s,-\beta-s)}} \\ &= (-J_n^{(\alpha,-\beta)}, -\mathcal{L}_{\alpha,\beta}^{2s}-J_m^{(\alpha,-\beta)})_{\omega^{(\alpha,-\beta)}} = \lambda_{n,s}^{(\beta,\alpha)}\gamma_n^{(\alpha,-\beta)}\delta_{nm}. \end{aligned} \quad (3.28)$$

Proof. We just prove the results for $+J_n^{(-\alpha,\beta)}(x)$. For $\alpha > 0$ and $s \in (0, 1)$, since $+J_n^{(-\alpha,\beta)}(1) = 0$, we find from (2.11) that D_+^s can be replaced by CD_+^s . Accordingly, (3.25) follows from (3.21) immediately.

We now show the fractional integration by parts can get through. By (2.30) and (3.24),

$$I_-^{1-s}\{\omega^{(-\alpha+s,\beta+s)}CD_+^s+J_n^{(-\alpha,\beta)}\} = \tilde{d}_{n,s}^{\alpha,\beta}(1+x)^{\beta+1}P_n^{(\alpha-1,\beta+1)}(x), \quad (3.29)$$

where the constant $\tilde{d}_{n,s}^{\alpha,\beta}$ can be worked out. Clearly, it vanishes at $x = -1$. On the other hand, $+J_m^{(-\alpha,\beta)}(1) = 0$. Therefore, we can perform the rule (2.12a) to obtain the second identity in (3.26). The orthogonality follows from (3.10) and (3.25).

The results for $-J_n^{(\alpha,-\beta)}(x)$ can be derived similarly. \square

3.3. Relation with Jacobi poly-fractonomials. In a very recent paper, Zayernouri and Karniadakis [30] introduced a family of Jacobi poly-fractonomials (JPFs) from the eigenfunctions of a singular fractional Sturm-Liouville problem. We first recall their definition.

Definition 3.2 (Jacobi poly-fractonomials [30]). For $\mu \in (0, 1)$, the Jacobi poly-fractonomials of order μ are defined as follows.

- For $-1 < \alpha < 2 - \mu$ and $-1 < \beta < \mu - 1$,

$${}^{(1)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x) = (1+x)^{\mu-(\beta+1)} P_{n-1}^{(\alpha+1-\mu, \mu-(\beta+1))}(x), \quad n \geq 1. \quad (3.30)$$

- For $-1 < \alpha < \mu - 1$ and $-1 < \beta < 2 - \mu$,

$${}^{(2)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x) = (1-x)^{\mu-(\alpha+1)} P_{n-1}^{(\mu-(\alpha+1), \beta+1-\mu)}(x), \quad n \geq 1. \quad (3.31)$$

As shown in [30, Thm. 4.2], the left JPFs are eigenfunctions of the singular fractional Sturm-Liouville equation:

$$D_+^\mu \{ \omega^{(\alpha+1, \beta+1)}(x) {}^C D_-^\mu \{ {}^{(1)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x) \} \} = {}^{(1)}\lambda_n^{(\alpha, \beta, \mu)} \omega^{(\alpha+1-\mu, \beta+1-\mu)}(x) {}^{(1)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x), \quad (3.32)$$

where

$${}^{(1)}\lambda_n^{(\alpha, \beta, \mu)} = \frac{\Gamma(n+\alpha+1)\Gamma(n+\mu-\beta-1)}{\Gamma(n-\beta-1)\Gamma(n-\mu+\alpha+1)}, \quad n \geq 1.$$

The right JPFs satisfy a similar equation.

It follows from (3.1)-(3.2) and (3.30)-(3.31) the relation:

$${}^{(1)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x) = -J_{n-1}^{(\alpha+1-\mu, \beta+1-\mu)}(x), \quad {}^{(2)}\mathcal{P}_n^{(\alpha, \beta, \mu)}(x) = +J_{n-1}^{(\alpha+1-\mu, \beta+1-\mu)}(x). \quad (3.33)$$

Observe that with the parameters $\{\mu, \alpha+1-\mu, \mu-(\beta+1)\}$ in place of $\{s, \alpha, \beta\}$ in (3.27), we obtain (3.32) exactly. However, the range of the parameters is $\alpha > -1$ and $-1 < \beta < 1-\mu$, so the condition on α is relaxed as opposite to that for (3.30). Indeed, the difference between the range of α is not surprising, as the GJFs here and JPFs in [30] are defined by different means.

4. APPROXIMATION BY GJFS

The main concern of this section is to show that approximation by GJF series leads to typical spectral convergence for functions in appropriate weighted Sobolev spaces involving fractional derivatives. Such approximation results play a crucial role in the analysis of spectral methods for fractional differential equations, see Section 5.

For simplicity of presentation, we only provide the detailed analysis for $\{+J_n^{(-\alpha, \beta)}\}$, as the results can be extended to $\{-J_n^{(\alpha, -\beta)}\}$ straightforwardly, thanks to (3.7). In the first place, we highlight some special GJFs of particular interest.

- For $\alpha > 0$ and $\beta \in \mathbb{R}$ (such that $-(n+\alpha+\beta) \notin \mathbb{N}$ and $-\beta \notin \mathbb{N}$), we have

$$D^l +J_n^{(-\alpha, \beta)}(1) = 0, \quad \text{for } l = 0, 1, \dots, [\alpha] - 1, \quad (4.1)$$

which naturally allows us to impose the one-sided boundary conditions: $u^{(l)}(1) = 0$ for $l = 0, 1, \dots, [\alpha] - 1$, and more importantly, it matches the singularity of the solution for prototypical fractional initial value problems, thanks to the fractional factor $(1-x)^\alpha$. Moreover, we can choose the parameter β (e.g., $\beta = -\alpha$) so that under the GJF basis, the linear systems of the fractional equations can be sparse and well-conditioned.

- For $\alpha > 0$ and $\beta = -[\alpha]$, we find from (3.5) that for $n \geq [\alpha]$,

$$D^l +J_n^{(-\alpha, -[\alpha])}(\pm 1) = 0, \quad \text{for } l = 0, 1, \dots, [\alpha] - 1, \quad (4.2)$$

which allows us to deal with two-sided boundary conditions: $u^{(l)}(\pm 1) = 0$, and to match the singularity of the solution to some prototypical fractional boundary value problems.

We introduce some notation to be used later. Let \mathcal{P}_N be the set of all algebraic (real-valued) polynomials of degree at most N . Let $\varpi(x) > 0$ for all $x \in \Lambda$, be a generic weight function. The weighted space $L^2_{\varpi}(\Lambda)$ is defined as in Admas [2] with the inner product and norm

$$(u, v)_{\varpi} = \int_{\Lambda} u(x)v(x)\varpi(x)dx, \quad \|u\|_{\varpi} = (u, u)_{\varpi}^{1/2}.$$

If $\varpi \equiv 1$, we omit the weight in the notation. In what follows, the Sobolev space $H^1(\Lambda)$ is also defined as usual.

4.1. Approximation results for GJFs $\{+J_n^{(-\alpha, \beta)}\}$. In view of the applications that we have in mind, we restrict the parameters to the set

$$+\Upsilon^{\alpha, \beta} := \{(\alpha, \beta) : \alpha > 0, \alpha + \beta > -1\}, \quad (4.3)$$

which we further split into three disjoint subsets:

$$\begin{aligned} +\Upsilon_1^{\alpha, \beta} &:= \{(\alpha, \beta) : \alpha > 0, \beta > -1\}; \\ +\Upsilon_2^{\alpha, \beta} &:= \{(\alpha, \beta) : \alpha > 0, -\alpha - 1 < \beta = -k \leq -1, k \in \mathbb{N}\}; \\ +\Upsilon_3^{\alpha, \beta} &:= \{(\alpha, \beta) : \alpha > 0, -\alpha - 1 < \beta < -1, -\beta \notin \mathbb{N}\}. \end{aligned} \quad (4.4)$$

4.1.1. Case I: $(\alpha, \beta) \in +\Upsilon_1^{\alpha, \beta} \cup +\Upsilon_2^{\alpha, \beta}$. Let us first consider $(\alpha, \beta) \in +\Upsilon_1^{\alpha, \beta}$. In this case, we define the finite-dimensional fractional-polynomial space:

$$+\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda) = \{\phi = (1-x)^{\alpha}\psi : \psi \in \mathcal{P}_N\} = \text{span}\{+J_n^{(-\alpha, \beta)} : 0 \leq n \leq N\}. \quad (4.5)$$

By the orthogonality (3.10), we can expand any $u \in L^2_{\omega^{(-\alpha, \beta)}}(\Lambda)$ as

$$u(x) = \sum_{n=0}^{\infty} \hat{u}_n^{(\alpha, \beta)} +J_n^{(-\alpha, \beta)}(x), \quad \text{where } \hat{u}_n^{(\alpha, \beta)} = \frac{1}{\gamma_n^{(\alpha, \beta)}} \int_{-1}^1 u +J_n^{(-\alpha, \beta)} \omega^{(-\alpha, \beta)} dx, \quad (4.6)$$

and there holds the Parseval identity:

$$\|u\|_{\omega^{(-\alpha, \beta)}}^2 = \sum_{n=0}^{\infty} \gamma_n^{(\alpha, \beta)} |\hat{u}_n^{(\alpha, \beta)}|^2. \quad (4.7)$$

Consider the $L^2_{\omega^{(-\alpha, \beta)}}$ -orthogonal projection upon $+\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda)$, defined by

$$(+\pi_N^{(-\alpha, \beta)} u - u, v_N)_{\omega^{(-\alpha, \beta)}} = 0, \quad \forall v_N \in +\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda). \quad (4.8)$$

By definition, we have

$$(+\pi_N^{(-\alpha, \beta)} u)(x) = \sum_{n=0}^N \hat{u}_n^{(\alpha, \beta)} +J_n^{(-\alpha, \beta)}(x). \quad (4.9)$$

We now consider $(\alpha, \beta) \in +\Upsilon_2^{\alpha, \beta}$. In this case, we modify (4.5) as

$$+\mathcal{F}_N^{(-\alpha, -k)}(\Lambda) = \{\phi = (1-x)^{\alpha}\psi : \psi \in \mathcal{P}_N \text{ such that } \psi^{(l)}(-1) = 0, 0 \leq l \leq k-1\}, \quad (4.10)$$

which incorporates the homogeneous boundary conditions at $x = -1$. Thanks to (3.5), we have

$$+\mathcal{F}_N^{(-\alpha, -k)}(\Lambda) = \text{span}\{+J_n^{(-\alpha, -k)}(x) : k \leq n \leq N\}. \quad (4.11)$$

In view of the orthogonality (3.11), we have the expansion like (4.6), that is, for any $u \in L^2_{\omega^{(-\alpha, -k)}}(\Lambda)$,

$$u(x) = \sum_{n=k}^{\infty} \hat{u}_n^{(\alpha, -k)} +J_n^{(-\alpha, -k)}(x), \quad \text{where } \hat{u}_n^{(\alpha, -k)} = \frac{1}{\gamma_n^{(\alpha, -k)}} \int_{-1}^1 u +J_n^{(-\alpha, -k)} \omega^{(-\alpha, -k)} dx, \quad (4.12)$$

so the identity (4.7) also holds for this expansion. The partial sum

$$+\pi_N^{(-\alpha, -k)} u(x) = \sum_{n=k}^N \hat{u}_n^{(\alpha, -k)} +J_n^{(-\alpha, -k)}(x), \quad (4.13)$$

is the $L^2_{\omega^{(-\alpha, -k)}}$ -orthogonal projection upon ${}^+\mathcal{F}_N^{(-\alpha, -k)}(\Lambda)$, namely,

$$({}^+\pi_N^{(-\alpha, -k)}u - u, v_N)_{\omega^{(-\alpha, -k)}} = 0, \quad \forall v_N \in {}^+\mathcal{F}_N^{(-\alpha, -k)}(\Lambda). \quad (4.14)$$

Remark 4.1. It is worthwhile to point out that for $(\alpha, \beta) \in {}^+\Upsilon_1^{\alpha, \beta} \cup {}^+\Upsilon_2^{\alpha, \beta}$, we have

$$(D_+^{\alpha+l}({}^+\pi_N^{(-\alpha, \beta)}u - u), D^l w_N)_{\omega^{(l, \alpha+\beta+l)}} = 0, \quad \forall w_N \in \mathcal{P}_N, \quad (4.15)$$

for all $l \in \mathbb{N}_0$. Notice that

$$({}^+\pi_N^{(-\alpha, \beta)}u - u)(x) = \sum_{n=N+1}^{\infty} \hat{u}_n^{(\alpha, \beta)} J_n^{(-\alpha, \beta)}(x),$$

and $\mathcal{P}_N = \text{span}\{P_n^{(0, \alpha+\beta)} : 0 \leq n \leq N\}$. Using the property $D_+^{\alpha+l} = (-1)^l D^l D_+^{\alpha}$, we obtain (4.15) from (3.14), (3.16) and the orthogonality of the classical Jacobi polynomials (cf. (2.24)). \square

To characterize the regularity of u , we introduce the non-uniformly weighted space involving fractional derivatives:

$${}^+\mathcal{B}_{\alpha, \beta}^m(\Lambda) := \{u \in L^2_{\omega^{(-\alpha, \beta)}}(\Lambda) : D_+^{\alpha+l}u \in L^2_{\omega^{(l, \alpha+\beta+l)}}(\Lambda) \text{ for } 0 \leq l \leq m\}, \quad m \in \mathbb{N}_0. \quad (4.16)$$

By (3.17) and (4.6) or (4.12), we have that for $(\alpha, \beta) \in {}^+\Upsilon_1^{\alpha, \beta} \cup {}^+\Upsilon_2^{\alpha, \beta}$ and $l \in \mathbb{N}_0$,

$$\|D_+^{\alpha+l}u\|_{\omega^{(l, \alpha+\beta+l)}}^2 = \sum_{n=\tilde{l}}^{\infty} h_{n, \tilde{l}}^{(\alpha, \beta)} |\hat{u}_n^{(\alpha, \beta)}|^2, \quad (4.17)$$

where $\tilde{l} = l$ for $(\alpha, \beta) \in {}^+\Upsilon_1^{\alpha, \beta}$; $\tilde{l} = \max\{l, k\}$ for $(\alpha, \beta) \in {}^+\Upsilon_2^{\alpha, \beta}$, and $h_{n, \tilde{l}}^{(\alpha, \beta)}$ is defined in (3.18).

Our main result on the projection errors for these two cases is stated as follows.

Theorem 4.1. Let $(\alpha, \beta) \in {}^+\Upsilon_1^{\alpha, \beta} \cup {}^+\Upsilon_2^{\alpha, \beta}$, and let $u \in {}^+\mathcal{B}_{\alpha, \beta}^m(\Lambda)$ with $m \in \mathbb{N}_0$.

- For $0 \leq l \leq m \leq N$,

$$\|D_+^{\alpha+l}({}^+\pi_N^{(-\alpha, \beta)}u - u)\|_{\omega^{(l, \alpha+\beta+l)}} \leq N^{(l-m)/2} \sqrt{\frac{(N-m+1)!}{(N-l+1)!}} \|D_+^{\alpha+m}u\|_{\omega^{(m, \alpha+\beta+m)}}. \quad (4.18)$$

In particular, if m is fixed, then

$$\|D_+^{\alpha+l}({}^+\pi_N^{(-\alpha, \beta)}u - u)\|_{\omega^{(l, \alpha+\beta+l)}} \leq cN^{l-m} \|D_+^{\alpha+m}u\|_{\omega^{(m, \alpha+\beta+m)}}. \quad (4.19)$$

- For $0 \leq m \leq N$, we also have the $L^2_{\omega^{(-\alpha, \beta)}}$ -estimates:

$$\|{}^+\pi_N^{(-\alpha, \beta)}u - u\|_{\omega^{(-\alpha, \beta)}} \leq cN^{-\alpha} \sqrt{\frac{(N-m+1)!}{(N+m+1)!}} \|D_+^{\alpha+m}u\|_{\omega^{(m, \alpha+\beta+m)}}. \quad (4.20)$$

In particular, if m is fixed, then

$$\|{}^+\pi_N^{(-\alpha, \beta)}u - u\|_{\omega^{(-\alpha, \beta)}} \leq cN^{-(\alpha+m)} \|D_+^{\alpha+m}u\|_{\omega^{(m, \alpha+\beta+m)}}. \quad (4.21)$$

Here, $c \approx 1$ for $N \gg 1$.

Proof. By (4.6) (or (4.12)), (4.8) (or (4.14)) and (4.17),

$$\begin{aligned} \|D_+^{\alpha+l}({}^+\pi_N^{(-\alpha, \beta)}u - u)\|_{\omega^{(l, \alpha+\beta+l)}}^2 &= \sum_{n=N+1}^{\infty} h_{n, l}^{(\alpha, \beta)} |\hat{u}_n^{(\alpha, \beta)}|^2 = \sum_{n=N+1}^{\infty} \frac{h_{n, l}^{(\alpha, \beta)}}{h_{n, m}^{(\alpha, \beta)}} h_{n, m}^{(\alpha, \beta)} |\hat{u}_n^{(\alpha, \beta)}|^2 \\ &\leq \frac{h_{N+1, l}^{(\alpha, \beta)}}{h_{N+1, m}^{(\alpha, \beta)}} \|D_+^{\alpha+m}u\|_{\omega^{(m, \alpha+\beta+m)}}^2. \end{aligned} \quad (4.22)$$

We now estimate the constant factor. By (2.14), (3.18) and a direct calculation, we find that for $0 \leq l \leq m \leq N$,

$$\begin{aligned} \frac{h_{N+1,l}^{(\alpha,\beta)}}{h_{N+1,m}^{(\alpha,\beta)}} &= \frac{\Gamma(N+\alpha+\beta+l+2)}{\Gamma(N+\alpha+\beta+m+2)} \frac{(N-m+1)!}{(N-l+1)!} \\ &= \frac{1}{(N+\alpha+\beta+2+l) \cdots (N+\alpha+\beta+1+m)} \frac{(N-m+1)!}{(N-l+1)!} \\ &\leq N^{l-m} \frac{(N-m+1)!}{(N-l+1)!}, \end{aligned} \quad (4.23)$$

where we used the fact: $\alpha + \beta > -1$. Therefore, the estimate (4.18) follows from (4.22)-(4.23) immediately.

We now turn to (4.19). Let us recall the property of the Gamma function (see [1, (6.1.38)]):

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} \exp\left(-x + \frac{\theta}{12x}\right), \quad \forall x > 0, \quad 0 < \theta < 1. \quad (4.24)$$

We can show that for any constant $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, $n+a > 1$ and $n+b > 1$ (see [33, Lemma 2.1]),

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \leq \nu_n^{a,b} n^{a-b}, \quad (4.25)$$

where

$$\nu_n^{a,b} = \exp\left(\frac{a-b}{2(n+b-1)} + \frac{1}{12(n+a-1)} + \frac{(a-b)^2}{n}\right). \quad (4.26)$$

Using the property $\Gamma(n+1) = n!$ and (4.25), we find that for $m \leq N$,

$$\frac{(N-m+1)!}{(N-l+1)!} \leq \nu_N^{2-m,2-l} N^{l-m}, \quad (4.27)$$

where $\nu_N^{2-m,2-l} \approx 1$ for fixed m and $N \gg 1$. Thus, we obtain (4.19) from (4.18) immediately.

The $L_{\omega^{(-\alpha,\beta)}}^2$ -estimates can be obtained by using the same argument. We sketch the derivation below. By (4.7) and (4.17),

$$\begin{aligned} \left\| {}^+\pi_N^{(-\alpha,\beta)} u - u \right\|_{\omega^{(-\alpha,\beta)}}^2 &= \sum_{n=N+1}^{\infty} \gamma_n^{(\alpha,\beta)} |\hat{u}_n^{(\alpha,\beta)}|^2 \\ &\leq \frac{\gamma_{N+1}^{(\alpha,\beta)}}{h_{N+1,m}^{(\alpha,\beta)}} \sum_{n=N+1}^{\infty} h_{n,m}^{(\alpha,\beta)} |\hat{u}_n^{(\alpha,\beta)}|^2 \leq \frac{\gamma_{N+1}^{(\alpha,\beta)}}{h_{N+1,m}^{(\alpha,\beta)}} \|D_+^{\alpha+m} u\|_{\omega^{(m,\alpha+\beta+m)}}^2. \end{aligned} \quad (4.28)$$

Working out the constants by (2.25) and (3.18), we use (4.25) again to get that

$$\begin{aligned} \frac{\gamma_{N+1}^{(\alpha,\beta)}}{h_{N+1,m}^{(\alpha,\beta)}} &= \frac{\Gamma(N+\beta+2)}{\Gamma(N+\alpha+2)} \frac{\Gamma(N+m+2)}{\Gamma(N+\alpha+\beta+m+2)} \frac{(N-m+1)!}{(N+m+1)!} \\ &\leq \nu_N^{\alpha+2,\beta+2} N^{\beta-\alpha} \nu_N^{2,\alpha+\beta+2} (N+m)^{-(\alpha+\beta)} \frac{(N-m+1)!}{(N+m+1)!} \\ &\leq cN^{-2(\alpha+m)} \quad (\text{if } m \text{ is fixed}). \end{aligned} \quad (4.29)$$

This ends the proof. \square

Remark 4.2. We see from the above estimates that optimal order of convergence can be attained for approximation of u by its orthogonal projection ${}^+\pi_N^{(-\alpha,\beta)} u$ in both $L_{\omega^{(-\alpha,\beta)}}^2(\Lambda)$ and ${}^+\mathcal{B}_{\alpha,\beta}^l(\Lambda)$, when u belongs to a properly weighted space involving proper orders of fractional derivatives. \square

4.1.2. *Case II:* $(\alpha, \beta) \in {}^+\Upsilon_3^{\alpha, \beta}$. In this case, the main difficulty resides in that the GJFs $\{+J_n^{(-\alpha, \beta)}\}$ are no longer orthogonal on Λ . Thus, we adopt a different route to derive the approximation results. For any u such that $D_+^\alpha u \in L_{\omega(0, \alpha+\beta)}^2(\Lambda)$, it admits the following unique expansion:

$$D_+^\alpha u(x) = \sum_{n=0}^{\infty} \hat{v}_n^{(0, \alpha+\beta)} P_n^{(0, \alpha+\beta)}(x), \quad (4.30)$$

where by (2.24),

$$\hat{v}_n^{(0, \alpha+\beta)} = \frac{1}{\gamma_n^{(0, \alpha+\beta)}} \int_{-1}^1 D_+^\alpha u(x) P_n^{(0, \alpha+\beta)}(x) (1+x)^{\alpha+\beta} dx. \quad (4.31)$$

From the definition of the usual orthogonal projection operator $\Pi_N^{(0, \alpha+\beta)} : L_{\omega(0, \alpha+\beta)}^2(\Lambda) \rightarrow \mathcal{P}_N$, we have

$$(\Pi_N^{(0, \alpha+\beta)}(D_+^\alpha u) - D_+^\alpha u, v_N)_{\omega(0, \alpha+\beta)} = 0, \quad \forall v_N \in \mathcal{P}_N, \quad (4.32)$$

and

$$\Pi_N^{(0, \alpha+\beta)}(D_+^\alpha u)(x) = \sum_{n=0}^N \hat{v}_n^{(0, \alpha+\beta)} P_n^{(0, \alpha+\beta)}(x). \quad (4.33)$$

Let ${}^+\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda)$ be the finite-dimensional space as defined in (4.5) but for $(\alpha, \beta) \in \Upsilon_3^{\alpha, \beta}$.

Lemma 4.1. *Let $\alpha > 0$, $-\alpha - 1 < \beta < -1$ and $-\beta \notin \mathbb{N}$. For any u such that $D_+^\alpha u \in L_{\omega(0, \alpha+\beta)}^2(\Lambda)$, there exists a unique $u_N =: {}^+\pi_N^{(-\alpha, \beta)} u \in {}^+\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda)$ such that*

$$\Pi_N^{(0, \alpha+\beta)}(D_+^\alpha u)(x) = D_+^\alpha ({}^+\pi_N^{(-\alpha, \beta)} u)(x), \quad (4.34)$$

and

$$(D_+^\alpha ({}^+\pi_N^{(-\alpha, \beta)} u - u), v_N)_{\omega(0, \alpha+\beta)} = 0, \quad \forall v_N \in \mathcal{P}_N. \quad (4.35)$$

Proof. From the expansion coefficients $\{\hat{v}_n^{(0, \alpha+\beta)}\}_{n=0}^N$ in (4.31), we construct

$$u_N(x) = \sum_{n=0}^N \frac{n! \hat{v}_n^{(0, \alpha+\beta)}}{\Gamma(n + \alpha + 1)} {}^+J_n^{(-\alpha, \beta)}(x) \in {}^+\mathcal{F}_N^{(-\alpha, \beta)}(\Lambda). \quad (4.36)$$

Acting D_+^α on both sides, we obtain from (3.14) and (4.33) that

$$D_+^\alpha u_N(x) = \sum_{n=0}^N \hat{v}_n^{(0, \alpha+\beta)} P_n^{(0, \alpha+\beta)}(x) = \Pi_N^{(0, \alpha+\beta)}(D_+^\alpha u)(x).$$

Note that the expansion in (4.31) is unique, so we specifically denote u_N by ${}^+\pi_N^{(-\alpha, \beta)} u$, and (4.34) is shown.

The property (4.35) is a direct consequence of (4.32) and (4.34). \square

With Lemma 4.1 at our disposal, we can obtain the following error estimates.

Theorem 4.2. *Let $\alpha > 0$, $-\alpha - 1 < \beta < -1$ and $-\beta \notin \mathbb{N}$, and let ${}^+\pi_N^{(-\alpha, \beta)}$ be defined in Lemma 4.1. Suppose that $D_+^{\alpha+l} u \in L_{\omega(l, \alpha+\beta+l)}^2(\Lambda)$ with $0 \leq l \leq m \leq N$. Then we have*

$$\|D_+^\alpha ({}^+\pi_N^{(-\alpha, \beta)} u - u)\|_{\omega(0, \alpha+\beta)} \leq N^{-m/2} \sqrt{\frac{(N-m+1)!}{(N+1)!}} \|D_+^{\alpha+m} u\|_{\omega(m, \alpha+\beta+m)}. \quad (4.37)$$

In particular, if m is fixed, we have

$$\|D_+^\alpha ({}^+\pi_N^{(-\alpha, \beta)} u - u)\|_{\omega(0, \alpha+\beta)} \leq c N^{-m} \|D_+^{\alpha+m} u\|_{\omega(m, \alpha+\beta+m)}, \quad (4.38)$$

where the constant $c \approx 1$ for $N \gg 1$.

Proof. Using the relation (4.34), we further derive from (2.24), (4.30) and (4.33) that

$$\begin{aligned} \|D_+^\alpha (\pi_N^{(-\alpha, \beta)} u - u)\|_{\omega^{(0, \alpha + \beta)}}^2 &= \|\Pi_N^{(0, \alpha + \beta)} (D_+^\alpha u) - D_+^\alpha u\|_{\omega^{(0, \alpha + \beta)}}^2 \\ &= \sum_{n=N+1}^{\infty} \gamma_n^{(0, \alpha + \beta)} |\hat{v}_n^{(0, \alpha + \beta)}|^2. \end{aligned} \quad (4.39)$$

We find from (2.24), (3.16) and (4.30) that

$$\|D_+^{\alpha+m} u\|_{\omega^{(m, \alpha + \beta + m)}}^2 = \sum_{n=m}^{\infty} \mu_{n,m}^{(0, \alpha + \beta)} |\hat{v}_n^{(0, \alpha + \beta)}|^2, \quad (4.40)$$

where for $n \geq m$,

$$\mu_{n,m}^{(0, \alpha + \beta)} = (\kappa_{n,m}^{(0, \alpha + \beta)})^2 \gamma_{n-m}^{(m, \alpha + \beta + m)} = \frac{2^{\alpha + \beta + 1} n! \Gamma(n + \alpha + \beta + m + 1)}{(2n + \alpha + \beta + 1)(n - m)! \Gamma(n + \alpha + \beta + 1)}. \quad (4.41)$$

In view of the above facts, we work out the constants by using (2.25) and obtain

$$\begin{aligned} \|D_+^\alpha (\pi_N^{(-\alpha, \beta)} u - u)\|_{\omega^{(0, \alpha + \beta)}}^2 &\leq \frac{\gamma_{N+1}^{(0, \alpha + \beta)}}{\mu_{N+1,m}^{(0, \alpha + \beta)}} \|D_+^{\alpha+m} u\|_{\omega^{(m, \alpha + \beta + m)}}^2 \\ &= \frac{1}{(N + \alpha + \beta + 2)_m} \frac{(N + 1 - m)!}{(N + 1)!} \|D_+^{\alpha+m} u\|_{\omega^{(m, \alpha + \beta + m)}}^2 \\ &\leq N^{-m} \frac{(N + 1 - m)!}{(N + 1)!} \|D_+^{\alpha+m} u\|_{\omega^{(m, \alpha + \beta + m)}}^2. \end{aligned} \quad (4.42)$$

This yields (4.37). For fixed m , we apply (4.25) to deal with the above factorials and derive (4.38) immediately. \square

4.2. Approximation results for GJFs $\{-J_n^{(\alpha, -\beta)}\}$. The results established in the previous subsection can be extended to $\{-J_n^{(\alpha, -\beta)}\}$ straightforwardly, thanks to (3.7). Below, we sketch the corresponding notation and results.

Define the parameter set

$$-\Upsilon^{\alpha, \beta} := \{(\alpha, \beta) : \beta > 0, \alpha + \beta > -1\}, \quad (4.43)$$

which we split into three disjoint subsets:

$$\begin{aligned} -\Upsilon_1^{\alpha, \beta} &:= \{(\alpha, \beta) : \beta > 0, \alpha > -1\}; \\ -\Upsilon_2^{\alpha, \beta} &:= \{(\alpha, \beta) : \beta > 0, -\beta - 1 < \alpha = -k \leq -1, k \in \mathbb{N}\}; \\ -\Upsilon_3^{\alpha, \beta} &:= \{(\alpha, \beta) : \beta > 0, -\beta - 1 < \alpha < -1, -\alpha \notin \mathbb{N}\}. \end{aligned} \quad (4.44)$$

Consider the $L_{\omega^{(\alpha, -\beta)}}^2$ -orthogonal projection: $-\pi_N^{(\alpha, -\beta)} u \in -\mathcal{F}_N^{(\alpha, -\beta)}(\Lambda)$ for $(\alpha, \beta) \in -\Upsilon_1^{\alpha, \beta} \cup -\Upsilon_2^{\alpha, \beta}$, where the notation is defined in a fashion similar to that in the previous subsection. In this context, we define

$$-\mathcal{B}_{\alpha, \beta}^m(\Lambda) := \{u \in L_{\omega^{(\alpha, -\beta)}}^2(\Lambda) : D_-^{\beta+l} u \in L_{\omega^{(\alpha + \beta + l, l)}}^2(\Lambda) \text{ for } 0 \leq l \leq m\}, \quad m \in \mathbb{N}_0. \quad (4.45)$$

Following the argument as in the proof of Theorem 4.1, we can derive the following error estimates.

Theorem 4.3. *Let $(\alpha, \beta) \in -\Upsilon_1^{\alpha, \beta} \cup -\Upsilon_2^{\alpha, \beta}$, and let $u \in -\mathcal{B}_{\alpha, \beta}^m(\Lambda)$ with $m \in \mathbb{N}_0$.*

- For $0 \leq l \leq m \leq N$,

$$\|D_-^{\beta+l} (-\pi_N^{(\alpha, -\beta)} u - u)\|_{\omega^{(\alpha + \beta + l, l)}} \leq N^{(l-m)/2} \sqrt{\frac{(N - m + 1)!}{(N - l + 1)!}} \|D_-^{\beta+m} u\|_{\omega^{(\alpha + \beta + m, m)}}. \quad (4.46)$$

In particular, if m is fixed, we have

$$\|D_-^{\beta+l}(-\pi_N^{(\alpha,-\beta)}u - u)\|_{\omega^{(\alpha+\beta+l,l)}} \leq cN^{l-m} \|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}. \quad (4.47)$$

- For $0 \leq m \leq N$, we also have the $L_{\omega^{(\alpha,-\beta)}}^2$ -estimates:

$$\|-\pi_N^{(\alpha,-\beta)}u - u\|_{\omega^{(\alpha,-\beta)}} \leq cN^{-\beta} \sqrt{\frac{(N-m+1)!}{(N+m+1)!}} \|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}. \quad (4.48)$$

In particular, if m is fixed, then

$$\|-\pi_N^{(\alpha,-\beta)}u - u\|_{\omega^{(\alpha,-\beta)}} \leq cN^{-(\beta+m)} \|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}. \quad (4.49)$$

Here, $c \approx 1$ for $N \gg 1$.

Next, we consider $(\alpha, \beta) \in -\Upsilon_3^{\alpha,\beta}$. For $D_-^\beta u \in L_{\omega^{(\alpha+\beta,0)}}^2(\Lambda)$, we define the operator $-\Pi_N^{(\alpha,-\beta)}$ similarly as that in Lemma 4.1. Following the lines as in the proof of Theorem 4.2, we can obtain the following estimates.

Theorem 4.4. Let $(\alpha, \beta) \in -\Upsilon_3^{\alpha,\beta}$. Suppose that $D_-^{\beta+l}u \in L_{\omega^{(\alpha+\beta+l,l)}}^2(\Lambda)$ with $0 \leq l \leq m \leq N$. Then we have

$$\|D_-^\beta(-\pi_N^{(\alpha,-\beta)}u - u)\|_{\omega^{(\alpha+\beta,0)}} \leq N^{-m/2} \sqrt{\frac{(N-m+1)!}{(N+1)!}} \|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}. \quad (4.50)$$

In particular, if m is fixed, we have

$$\|D_-^\beta(-\pi_N^{(\alpha,-\beta)}u - u)\|_{\omega^{(\alpha+\beta,0)}} \leq cN^{-m} \|D_-^{\beta+m}u\|_{\omega^{(\alpha+\beta+m,m)}}, \quad (4.51)$$

where the constant $c \approx 1$ for $N \gg 1$.

Remark 4.3. To have a better understanding of the above approximation results, we compare GJF and Legendre approximation to the function:

$$u(x) = (1+x)^b g(x), \quad b \in \mathbb{R}^+, \quad x \in \Lambda, \quad (4.52)$$

where g is analytic within a domain containing Λ . Recall the best L^2 -approximation of u by its orthogonal projection $\pi_N^L u$ (see, e.g., [25, Ch. 3]):

$$\|\pi_N^L u - u\| \leq cN^{1-m} \|D^m u\|_{\omega^{(m,m)}}.$$

If b is non-integer, a direct calculation shows that u has a limited regularity: $m < 1 + 2b - \epsilon$ for small $\epsilon > 0$, in this usual weighted norm involving ordinary derivatives. We now consider GJF approximation (4.48) to u in (4.52). Using the explicit formulas for fractional integral/derivative of $(1+x)^b$ and the Leibniz' formula (see [6, Ch. 2]), we find that if $\beta = b$, $D_-^{\beta+m}u$ is analytic as well for any $m \in \mathbb{N}_0$, so by (4.52) with $\alpha = 0, \beta = b$ and $m = N$, and using (4.24), we have

$$\|-\pi_N^{(0,-\beta)}u - u\| \leq \|-\pi_N^{(0,-\beta)}u - u\|_{\omega^{(0,-\beta)}} \leq cN^{-(\beta+1/4)} \left(\frac{e}{2N}\right)^N \|D_-^{\beta+N}u\|_{\omega^{(\beta+N,N)}}.$$

This implies the exponential convergence $O(e^{-cN})$.

Also note that if u is smooth, e.g., $b \in \mathbb{N}$, we can only get a limited convergence rate by choosing a non-integer β . Indeed, a direct calculation by using the formulas in [6] yields

$$D_-^{\beta+m}u = (1+x)^{b-\beta-m} h(x),$$

where h is analytic. Therefore, we have that $\|D_-^{\beta+m}u\|_{\omega^{(\beta+m,m)}} < \infty$ only when $m + 2\beta < 1 + 2b - \epsilon$. \square

5. APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS

It is well-known that the underlying solution of a FDE usually exhibits singular behaviors at the boundaries, even when the given data are regular. Accordingly, the solution and data are not always in the same types of Sobolev spaces as opposite to DEs of integer derivatives. Hence, the use of polynomial approximations can only lead to limited convergence rate. In this section, we shall construct Petrov-Galerkin spectral methods using GJFs as basis functions for several prototypical FDEs, and demonstrate that

- (i) The convergence rate of our approach only depends on the regularity of the data in the usual weighted Sobolev space, regardless of the singular behavior of their solutions, so truly spectral accuracy can be achieved, if the input of a FDE is smooth enough.
- (ii) With a suitable choice of the parameters in the GJF basis, the resulted linear systems are usually sparse and sometimes diagonal.

We shall provide ample numerical results to validate the theoretical analysis. We believe that the study of these prototypical FDEs can shed light on the investigation of more complicated FDEs.

5.1. Fractional initial value problems (FIVPs). As a first example, we consider the fractional initial value problem of order $s \in (k-1, k)$ with $k \in \mathbb{N}$:

$$D_+^s u(x) = f(x), \quad x \in \Lambda; \quad u^{(l)}(1) = 0, \quad l = 0, \dots, k-1, \quad (5.1)$$

where $f \in L^2(\Lambda)$.

The GJF-spectral-Petrov-Galerkin scheme is to find $u_N \in {}^+\mathcal{F}_N^{(-s, -s)}(\Lambda)$ (defined in (4.5)) such that

$$(D_+^s u_N, v_N) = (f, v_N), \quad \forall v_N \in \mathcal{P}_N. \quad (5.2)$$

Using the GJF basis, we can write

$$u_N(x) = \sum_{n=0}^N \tilde{u}_n^{(s)} J_n^{(-s, -s)}(x) \in {}^+\mathcal{F}_N^{(-s, -s)}(\Lambda). \quad (5.3)$$

Taking $v_N = P_k$ in (5.2), we derive from (3.14) and the orthogonality of Legendre polynomials that

$$\tilde{u}_n^{(s)} = \frac{n!}{\Gamma(n+s+1)} \tilde{f}_n, \quad 0 \leq n \leq N, \quad (5.4)$$

where \tilde{f}_n is the $(n+1)$ -th coefficient of the Legendre expansion of f . Therefore, we obtain the numerical solution u_N by inserting (5.4) into (5.3).

The following error estimate shows the spectral accuracy of this GJF-Petrov-Galerkin approximation.

Theorem 5.1. *Let u and u_N be the solution of (5.1) and (5.2), respectively. If $f^{(l)} \in L_{\omega^{(l,l)}}^2(\Lambda)$ for all $0 \leq l \leq m$, then we have that for $0 \leq m \leq N$,*

$$\|D_+^s(u - u_N)\| \leq cN^{-m} \|f^{(m)}\|_{\omega^{(m,m)}}, \quad (5.5)$$

where c is a positive constant independent of u, N and m .

Proof. Let ${}^+\pi_N^{(-s, -s)}u$ be the same as in (4.14) for $0 < s < 1$, and as in (4.35) for $s > 1$, respectively. By (4.15) (with $l = 0$) and (4.35), we have

$$(D_+^s({}^+\pi_N^{(-s, -s)}u - u), \psi) = 0, \quad \forall \psi \in \mathcal{P}_N.$$

Then by (5.1),

$$(f - D_+^s({}^+\pi_N^{(-s, -s)}u), \psi) = (D_+^s u - D_+^s({}^+\pi_N^{(-s, -s)}u), \psi) = 0, \quad \forall \psi \in \mathcal{P}_N. \quad (5.6)$$

Let $\pi_N f$ be the L^2 -orthogonal projection of f upon \mathcal{P}_N . We infer from (5.6) that $D_+^s({}^+\pi_N^{(-s, -s)}u) = \pi_N f$. On the other hand, by (5.2), $D_+^s u_N = \pi_N f$. Thus, we have $D_+^s({}^+\pi_N^{(-s, -s)}u - u_N) = 0$.

Therefore, it follows from Theorem 4.1 (with $\alpha = -\beta = s$ and $0 < s < 1$), Theorem 4.2 (with $\alpha = -\beta = s$ and $s > 1$) that

$$\|D_+^s(u - u_N)\| = \|D_+^s(u - {}^+\pi_N^{(-s, -s)}u)\| \leq cN^{-m}\|D_+^{s+m}u\|_{\omega(m, m)} \leq cN^{-m}\|f^{(m)}\|_{\omega(m, m)}.$$

This ends the proof. \square

Remark 5.1. One can also construct a similar Petrov-Galerkin scheme for the following more general FIVPs of order $s \in (k-1, k)$ with $k \in \mathbb{N}$:

$$\begin{aligned} \mathcal{L}[u] &:= D_+^s u(x) + p_1(x)D_+^{s-1}u(x) + \cdots + p_{k-1}(x)D_+^{s-k+1}u(x) = f(x), \quad x \in \Lambda; \\ u^{(l)}(1) &= 0, \quad l = 0, \dots, k-1, \end{aligned} \quad (5.7)$$

where f and $\{p_j\}$ are continuous functions on $\bar{\Lambda}$. We find from (3.12) that $\mathcal{L}[J_n^{(-s, s)}]$ is a combination of products of p_j and polynomials. Hence, one can derive spectrally accurate error estimates as in Theorem 5.1. If $\{p_j\}$ are constants, the corresponding linear system will be sparse; for general $\{p_j\}$, one can use a preconditioned iterative algorithm as in the integer s case by using the problem with suitable constant constants as a preconditioner (cf. [25]). \square

5.2. Fractional boundary value problems (FBVPs). In accordance with usual BVPs, it is necessary to classify a FBVP of order ν as even or odd order as follows.

- If $\nu = s + k$ with $s \in (k-1, k)$ and $k \in \mathbb{N}$, we say it is of even order. In this case, $2k$ boundary conditions should be imposed.
- If $\nu = s + k$ with $s \in (k, k+1)$ and $k \in \mathbb{N}$, we say it is odd order. In this case, $2k+1$ boundary conditions should be imposed.

In practice, the boundary conditions can be of integral type or usual Dirichlet type, which often-times lead to different singular behaviour of the solution and should be treated quite differently. For easy of accessibility, we first consider FBVPs with integral boundary conditions (BCs), and then discuss the more complicated Dirichlet BCs.

5.2.1. FBVPs with integral BCs. To fix the idea, we consider the fractional boundary value problem of order $\nu \in (1, 2)$:

$$D_+^\nu u(x) = f(x), \quad x \in \Lambda; \quad I_+^\mu u(\pm 1) = 0, \quad (5.8)$$

where $\mu := 2 - \nu \in (0, 1)$, and I_+^μ is the fractional integral operator defined in (2.2). Here, $f(x)$ is a given function with regularity to be specified later.

Let $H_0^1(\Lambda) = \{u \in H^1(\Lambda) : u(\pm 1) = 0\}$, and $H^{-1}(\Lambda)$ be its dual space. Using the property: $D_+^\nu = D_+^2 I_+^\mu$ (see (2.8)), we can formulate the weak form of (5.31) as: Find $v := I_+^\mu u \in H_0^1(\Lambda)$ such that

$$(Dv, Dw) = (f, w), \quad \forall w \in H_0^1(\Lambda). \quad (5.9)$$

It is well-known that for any $f \in H^{-1}(\Lambda)$, it admits a unique solution $v \in H_0^1(\Lambda)$. Then we can recover u uniquely from $u = D_+^\mu v$, thanks to (2.9).

As already mentioned, it is important to understand the singular behavior of the solution so as to compass the choice of the parameter that can match the singularity. For this purpose, we act I_+^2 on both sides of (5.8) and impose the boundary conditions, leading to

$$I_+^\mu u(x) = -I_+^2 f(x) + \frac{I_+^2 f(-1)}{2}(1-x). \quad (5.10)$$

Thus, by (2.9),

$$u(x) = D_+^\mu I_+^\mu u(x) = -I_+^\nu f(x) + \frac{I_+^2 f(-1)}{2\Gamma(2-\mu)}(1-x)^{1-\mu}. \quad (5.11)$$

Correspondingly, we define the finite-dimensional fractional-polynomial solution space:

$$V_N := \{\phi = (1-x)^{1-\mu}\psi : \psi \in \mathcal{P}_{N-1} \text{ such that } I_+^\mu \phi(-1) = 0\}. \quad (5.12)$$

The GJF-Petrov-Galerkin approximation is to find $u_N \in V_N$ such that

$$(D_+^{1-\mu} u_N, Dw_N) = (f, w_N), \quad \forall w_N \in \mathcal{P}_N^0 := \mathcal{P}_N \cap H_0^1(\Lambda). \quad (5.13)$$

In terms of error analysis, it is more convenient to formulate (5.13) into an equivalent Galerkin approximation (see (5.18) below). Indeed, note that

$$\mathcal{P}_N = \text{span}\{P_n^{(1-\mu, \mu-1)} : 0 \leq n \leq N\}, \quad (5.14)$$

and by (2.29) with $\rho = \mu, \alpha = 1 - \mu$ and $\beta = \mu - 1$,

$$I_+^\mu + J_n^{(\mu-1, \mu-1)}(x) = \frac{\Gamma(n+2-\mu)}{(n+1)!} + J_n^{(-1, -1)}(x) = \frac{\Gamma(n+2-\mu)}{n!} \int_x^1 P_n(y) dy, \quad (5.15)$$

where we used the formula derived from integrating the Sturm-Liouville equation of Legendre polynomials and using (3.16):

$$I_+^1 P_n(x) = \int_x^1 P_n(y) dy = \frac{1}{2n} (1-x^2) P_{n-1}^{(1,1)}(x) = \frac{1}{n+1} + J_n^{(-1, -1)}(x), \quad n \geq 1. \quad (5.16)$$

Since for $n \geq 1, I_+^\mu + J_n^{(\mu-1, \mu-1)}(\pm 1) = 0$, we have

$$V_N = \text{span}\{+J_n^{(\mu-1, \mu-1)} : 1 \leq n \leq N-1\}; \quad \mathcal{P}_N^0 = \text{span}\{I_+^1 P_n : 1 \leq n \leq N-1\}. \quad (5.17)$$

Thus, we infer from (5.15) that the operator I_+^μ is an isomorphism between V_N and \mathcal{P}_N^0 . Then we can equivalently formulate (5.13) as: Find $v_N := I_+^\mu u_N \in \mathcal{P}_N^0$ such that

$$(Dv_N, Dw_N) = (f, w_N), \quad \forall w_N \in \mathcal{P}_N^0, \quad (5.18)$$

which admits a unique solution as with (5.9). In fact, this formulation facilitates the error analysis, which can be accomplished by a standard argument.

Theorem 5.2. *Let u and u_N be the solution of (5.9) and (5.18), respectively. If $I_+^\mu u \in H_0^1(\Lambda)$ and $(1-x^2)^{(m-1)/2} D_+^{m-\mu} u \in L^2(\Lambda)$ with $m \in \mathbb{N}$, then we have*

$$\|D_+^{1-\mu}(u - u_N)\| \leq cN^{1-m} \|D_+^{m-\mu} u\|_{\omega^{(m-1, m-1)}}. \quad (5.19)$$

In particular, if $f^{(m-2)} \in L_{\omega^{(m-1, m-1)}}^2(\Lambda)$ with $m \geq 2$, we have

$$\|D_+^{1-\mu}(u - u_N)\| \leq cN^{1-m} \|f^{(m-2)}\|_{\omega^{(m-1, m-1)}}. \quad (5.20)$$

Here, c is a positive constant independent of N and u .

Proof. Using a standard argument for error analysis of Galerkin approximation, we find from (5.9) and (5.18) that

$$\|D(v - v_N)\| = \inf_{v_N^* \in \mathcal{P}_N^0} \|D(v - v_N^*)\|. \quad (5.21)$$

Let $\pi_N^{1,0}$ be the usual H_0^1 -orthogonal projection upon \mathcal{P}_N^0 , and recall the approximation result (see e.g., [25, Ch. 3]):

$$\|D(v - \pi_N^{1,0} v)\| \leq cN^{1-m} \|D^m v\|_{\omega^{(m-1, m-1)}}. \quad (5.22)$$

Recall that $v = I_+^\mu u$ and $v_N = I_+^\mu u_N$, so we take $v_N^* = \pi_N^{1,0} v$ in (5.21) and obtain the desired estimate (5.19) from (5.22).

The estimate (5.20) follows immediately from (5.8) and (5.19) by noting that $D^{m-2} f = D_+^{m-\mu} u$. \square

Now, we briefly describe the implementation of the scheme (5.13). Setting

$$u_N(x) = \sum_{n=1}^{N-1} \hat{u}_n + J_n^{(\mu-1, \mu-1)}(x), \quad f_j = (f, I_+^1 P_j), \quad 1 \leq j \leq N-1,$$

we find from (5.15) and the orthogonality of Legendre polynomials that

$$(D_+^{1-\mu} + J_n^{(\mu-1, \mu-1)}, DI_+^1 P_j) = \frac{\Gamma(n+2-\mu)}{n!} \frac{2n+1}{2} \delta_{jn}. \quad (5.23)$$

Then we obtain from (5.13) that

$$\hat{u}_n = \frac{2(n!)f_n}{(2n+1)\Gamma(n+2-\mu)}, \quad 1 \leq n \leq N-1. \quad (5.24)$$

We see that using the GJFs as basis functions, the matrix of the linear system is diagonal.

Remark 5.2. The above approach can be applied to higher-order FBVPs. For example, we consider the FBVP of “odd” order: $\nu = 3 - \mu$ with $\mu \in (0, 1)$:

$$D_+^\nu u(x) = f(x), \quad x \in \Lambda; \quad I_+^\mu u(\pm 1) = (I_+^\mu u)'(1) = 0. \quad (5.25)$$

To avoid repetition, we just outline the numerical scheme and implementation. Define the solution and test function spaces

$$\begin{aligned} V_N &:= \{\phi = (1-x)^{2-\mu}\psi : \psi \in \mathcal{P}_{N-2} \text{ such that } I_+^\mu \phi(-1) = 0\}, \\ V_N^* &:= \{\psi \in \mathcal{P}_N : \psi(\pm 1) = \psi'(-1) = 0\}. \end{aligned} \quad (5.26)$$

The GJF-Petrov-Galerkin scheme is to find $u_N \in V_N$ such that

$$(D_+^{2-\mu} u_N, Dw_N) = -(f, w_N), \quad \forall w_N \in V_N^*. \quad (5.27)$$

Using (2.29) with $\rho = \mu$, $\alpha = 2 - \mu$ and $\beta = \mu - 1$, we obtain from (2.22) that

$$I_+^\mu + J_n^{(\mu-2, \mu-1)}(x) = \frac{\Gamma(n+3-\mu)}{(n+2)!} J_n^{(-2, -1)}(x); \quad I_+^\mu + J_n^{(\mu-2, \mu-1)}(-1) = 0, \quad n \geq 1. \quad (5.28)$$

Hence, we have

$$V_N = \text{span}\{J_n^{(\mu-2, \mu-1)} : 1 \leq n \leq N-2\}, \quad V_N^* = \text{span}\{J_n^{(-1, -2)} : 1 \leq n \leq N-2\}. \quad (5.29)$$

By (2.5),

$$D_+^{2-\mu} + J_n^{(\mu-2, \mu-1)}(x) = \frac{\Gamma(n+3-\mu)}{n!} P_n^{(0,1)}(x), \quad D^- J_m^{(-1, -2)}(x) = (m+2)(1+x)P_m^{(0,1)}(x), \quad (5.30)$$

so by the orthogonality of the Jacobi polynomials $\{P_n^{(0,1)}\}$, the matrix of the system (5.27) is diagonal. \square

5.2.2. FBVPs with Dirichlet boundary conditions. Now, we turn to a more complicated case, and consider the fractional boundary value problem of even order $\nu = s + k$ with $s \in (k-1, k)$ and $k \in \mathbb{N}$:

$$D_+^\nu u(x) = f(x), \quad x \in \Lambda; \quad u^{(l)}(\pm 1) = 0, \quad l = 0, 1, \dots, k-1, \quad (5.31)$$

where $f(x)$ is a given function with regularity to be specified later.

We introduce the solution and test function spaces:

$$\begin{aligned} U &:= \{u \in L_{\omega^{(-s, -k)}}^2(\Lambda) : D_+^s u \in L_{\omega^{(0, s-k)}}^2(\Lambda)\}; \\ V &:= \{v \in L_{\omega^{(-k, -s)}}^2(\Lambda) : D^k v \in L_{\omega^{(0, k-s)}}^2(\Lambda)\}, \end{aligned} \quad (5.32)$$

equipped with the norms

$$\|u\|_U = (\|u\|_{\omega^{(-s, -k)}}^2 + \|D_+^s u\|_{\omega^{(0, s-k)}}^2)^{1/2}; \quad \|v\|_V = (\|v\|_{\omega^{(-k, -s)}}^2 + \|D^k v\|_{\omega^{(0, k-s)}}^2)^{1/2}. \quad (5.33)$$

For $u \in U$ and $v \in V$, we write

$$\begin{aligned} u(x) &= \sum_{n=k}^{\infty} \hat{u}_n {}^+J_n^{(-s, -k)}(x) = (1-x)^s(1+x)^k \sum_{n=k}^{\infty} \tilde{u}_n P_{n-k}^{(s, k)}(x); \\ v(x) &= \sum_{n=k}^{\infty} \hat{v}_n {}^-J_n^{(-k, -s)}(x) = (1-x)^k(1+x)^s \sum_{n=k}^{\infty} \tilde{v}_n P_{n-k}^{(k, s)}(x), \end{aligned} \quad (5.34)$$

where by (3.5), $\tilde{u}_n = 2^{-k} d_n^{k, s} \hat{u}_n$ and $\tilde{v}_n = (-1)^k 2^{-k} d_n^{k, s} \hat{v}_n$.

With the above setup, we can build in the homogenous boundary conditions and also perform fractional integration by parts (cf. Lemma 2.2). Hence, a weak form of (5.31) is to find $u \in U$ such that

$$a(u, v) := (D_+^s u, D^k v) = (f, v), \quad \forall v \in V. \quad (5.35)$$

Let ${}^+\mathcal{F}_N^{(-s, -k)}(\Lambda)$ and ${}^-\mathcal{F}_N^{(-k, -s)}(\Lambda)$ be the finite-dimensional spaces as defined in the previous section. Then the GJF-Petrov-Galerkin scheme for (5.35) is to find $u_N \in {}^+\mathcal{F}_N^{(-s, -k)}(\Lambda)$ such that

$$a(u_N, v_N) = (D_+^s u_N, D^k v_N) = (f, v_N), \quad \forall v_N \in {}^-\mathcal{F}_N^{(-k, -s)}(\Lambda). \quad (5.36)$$

We next show the unique solvability of (5.35)-(5.36) by verifying the Babuška-Brezzi inf-sup condition of the involved bilinear form. For this purpose, we first show the following equivalence of the norms.

Lemma 5.1. *Let $s \in (k-1, k)$, $k \in \mathbb{N}$, and U, V be the space defined in (5.32) and (5.33), respectively. Then we have*

$$\begin{aligned} C_{1,s} \|u\|_U &\leq \|D_+^s u\|_{\omega^{(0, s-k)}} \leq \|u\|_U, \quad \forall u \in U; \\ C_{2,s} \|v\|_V &\leq \|D^k v\|_{\omega^{(0, k-s)}} \leq \|v\|_V, \quad \forall v \in V, \end{aligned} \quad (5.37)$$

where

$$C_{1,s} = \left(1 + \frac{k!}{\Gamma(k+s+1)\Gamma(s+1)}\right)^{-1/2}; \quad C_{2,s} = \left(1 + \frac{\Gamma(s+1)}{k!\Gamma(k+s+1)}\right)^{-1/2}. \quad (5.38)$$

Proof. Given the expansion in (5.34), we derive from (3.11) and (3.17) that

$$\|u\|_{\omega^{(-s, -k)}}^2 = \sum_{n=k}^{\infty} \gamma_n^{(s, -k)} |\hat{u}_n|^2; \quad \|D_+^s u\|_{\omega^{(0, s-k)}}^2 = \sum_{n=k}^{\infty} h_{n,0}^{(s, -k)} |\hat{u}_n|^2, \quad (5.39)$$

where by (3.18),

$$h_{n,0}^{(s, -k)} = \frac{\Gamma^2(n+s+1)}{(n!)^2} \gamma_n^{(0, s-k)}. \quad (5.40)$$

Therefore,

$$\|u\|_{\omega^{(-s, -k)}}^2 = \sum_{n=k}^{\infty} \frac{\gamma_n^{(s, -k)}}{h_{n,0}^{(s, -k)}} h_{n,0}^{(s, -k)} |\hat{u}_n|^2 \leq \frac{\gamma_k^{(s, -k)}}{h_{k,0}^{(s, -k)}} \|D_+^s u\|_{\omega^{(0, s-k)}}^2,$$

so by (2.25), (5.33) and (5.40),

$$\|u\|_U^2 \leq \left(1 + \frac{\gamma_k^{(s, -k)}}{h_{k,0}^{(s, -k)}}\right) \|D_+^s u\|_{\omega^{(0, s-k)}}^2 = \frac{1}{C_{1,s}^2} \|D_+^s u\|_{\omega^{(0, s-k)}}^2.$$

This yields the first equivalence relation in (5.37).

Next, we find from (2.7) and (3.13) that

$$D^k \{ {}^-J_n^{(-k, -s)}(x) \} = \frac{\Gamma(n+s+1)}{\Gamma(n+s-k+1)} {}^-J_n^{(0, k-s)}(x), \quad (5.41)$$

so we have from the orthogonality (3.10)-(3.11) and (5.34) that

$$\|v\|_{\omega(-k,-s)}^2 = \sum_{n=k}^{\infty} |\hat{v}_n|^2 \gamma_n^{(s,-k)}; \quad \|D^k v\|_{\omega(0,k-s)}^2 = \sum_{n=k}^{\infty} q_n^{(s,k)} |\hat{v}_n|^2, \quad (5.42)$$

where

$$q_n^{(s,k)} := \frac{\Gamma^2(n+s+1)}{\Gamma^2(n+s-k+1)} \gamma_n^{(0,s-k)}. \quad (5.43)$$

Working out the constants leads to

$$\|v\|_{\omega(-k,-s)}^2 \leq \frac{\gamma_k^{(s,-k)}}{q_k^{(s,k)}} \|D^k v\|_{\omega(0,k-s)}^2 \leq \frac{\Gamma(s+1)}{k! \Gamma(k+s+1)} \|D^k v\|_{\omega(0,k-s)}^2. \quad (5.44)$$

Then by (5.33), the second equivalence follows immediately. \square

With the aid of Lemma 5.1, we can show the well-posedness of the weak form (5.35) and the Petrov-Galerkin scheme (5.36).

Theorem 5.3. *Let $f \in L_{\omega(s,k)}^2(\Lambda)$. Then the problem (5.35) admits a unique solution $u \in U$, and the scheme (5.36) admits a unique solution $u_N \in {}^+\mathcal{F}_N^{(-s,-k)}(\Lambda)$.*

Proof. It is clear that we have the continuity of the bilinear form on $U \times V$:

$$|a(u, v)| \leq \|u\|_U \|v\|_V, \quad \forall u \in U, \quad \forall v \in V. \quad (5.45)$$

The main task is to verify the inf-sup condition, that is, for any $0 \neq u \in U$,

$$\sup_{0 \neq v \in V} \frac{|a(u, v)|}{\|u\|_U \|v\|_V} \geq \eta := C_{1,s} C_{2,s}, \quad (5.46)$$

where $C_{1,s}$ and $C_{2,s}$ are given in (5.38). For this purpose, we construct $v_* \in V$ from the expansion of $u \in U$ in (5.34):

$$v_*(x) := \sum_{n=k}^{\infty} \hat{v}_n^* J_n^{(-k,-s)}(x) \quad \text{with} \quad \hat{v}_n^* = \frac{\Gamma(n+s-k+1)}{n!} \hat{u}_n. \quad (5.47)$$

By construction, one verifies by using from the orthogonality (2.24), (5.39) and (5.42) that

$$a(u, v_*) = \|D_+^s u\|_{\omega(0,s-k)}^2 = \|D^k v_*\|_{\omega(0,k-s)}^2. \quad (5.48)$$

Thus, using Lemma 5.1, we infer that for any $0 \neq u \in U$, there exists $0 \neq v_* \in V$ such that

$$a(u, v_*) = \|D_+^s u\|_{\omega(0,s-k)} \|D^k v_*\|_{\omega(0,k-s)} \geq C_{1,s} C_{2,s} \|u\|_U \|v_*\|_V. \quad (5.49)$$

This implies (5.46).

It remains to verify the “transposed” inf-sup condition:

$$\sup_{0 \neq u \in U} |a(u, v)| > 0, \quad \forall 0 \neq v \in V. \quad (5.50)$$

It can be shown by a converse process. In fact, assuming that $0 \neq v_* \in V$ is an arbitrary function, we construct

$$u(x) = \sum_{n=k}^{\infty} \hat{u}_n {}^+J_n^{(-s,-k)}(x) \quad \text{with} \quad \hat{u}_n = \frac{n!}{\Gamma(n+s-k+1)} \hat{v}_n^*.$$

Then we can derive (5.50) using (5.48).

Finally, if $f \in L_{\omega(s,k)}^2(\Lambda)$, we obtain from the Cauchy-Schwarz inequality that

$$|(f, v)| \leq \|f\|_{\omega(k,s)} \|v\|_{\omega(-k,-s)} \leq \|f\|_{\omega(k,s)} \|v\|_V.$$

Therefore, we claim from the Babuška-Brezzi theorem (cf. [4]) that the problem (5.2) has a unique solution.

Note that the inf-sup condition (5.46) is also valid for the discrete problem (5.36), which therefore admits a unique solution. \square

With the help of the above results, we can follow a standard argument to carry out the error analysis.

Theorem 5.4. *Let $s \in (k-1, k)$ with $k \in \mathbb{N}$, and let u and u_N be the solutions of (5.35) and (5.36), respectively. If $u \in U \cap \mathcal{B}_{s,-k}^m(\Lambda)$ with $0 \leq m \leq N$, then we have the error estimates:*

$$\|u - u_N\|_U \leq cN^{-m} \|D_+^{s+m} u\|_{\omega(m, s-k+m)}. \quad (5.51)$$

In particular, if $f^{(m-k)} \in L_{\omega(m, s-k+m)}^2(\Lambda)$ for $m \geq k$, we have

$$\|u - u_N\|_U \leq cN^{-m} \|f^{(m-k)}\|_{\omega(m, s-k+m)}. \quad (5.52)$$

Here, c is a positive constant independent of u, N and m .

Proof. Thanks to the inf-sup condition derived in the proof of the previous theorem, we have

$$\|u - u_N\|_U \leq (1 + \eta^{-1}) \|u - \phi\|_U, \quad \forall \phi \in {}^+\mathcal{F}_N^{(-s, -k)}(\Lambda), \quad (5.53)$$

where η is the inf-sup constant in (5.46). Let ${}^+\pi_N^{(-s, -k)}$ be the orthogonal projection operator as defined in (4.13)-(4.14). Taking $\phi = {}^+\pi_N^{(-s, -k)} u$ in (5.53), we obtain from Theorem 4.1 and Lemma 5.1 that

$$\begin{aligned} \|u - u_N\|_U &\leq (1 + \eta^{-1}) \|u - {}^+\pi_N^{(-s, -k)} u\|_U \\ &\leq (1 + \eta^{-1}) (C_{1,s})^{-1} \|D_+^s (u - {}^+\pi_N^{(-s, -k)} u)\|_{\omega(0, s-k)} \\ &\leq cN^{-m} \|D_+^{s+m} u\|_{\omega(m, s-k+m)}. \end{aligned} \quad (5.54)$$

This yields (5.51).

From the original equation (5.31), we obtain $D_+^\nu u = D_+^{s+k} u = f$, so (5.52) follows from (5.51) immediately. \square

Remark 5.3. By using a similar procedure as above, we can also construct a spectral Petrov-Galerkin method for the odd order FBVP of order $\nu = s + k$ and $s \in (k, k+1)$ with $k \in \mathbb{N}$:

$$D_+^\nu u(x) = f(x), \quad x \in \Lambda; \quad u^{(l)}(\pm 1) = 0, \quad l = 0, 1, \dots, k-1; \quad u^{(k)}(1) = 0, \quad (5.55)$$

and analyze the error as in Theorem 5.4. \square

5.3. Numerical results. In what follows, we provide some numerical results to illustrate the accuracy of the proposed GJF-Petrov-Galerkin schemes and to validate our error analysis. We give examples for two typical situations, that is, the source term $f(x)$ is smooth (so the solution $u(x)$ is singular), and vice versa. We examine the errors measured in both L^2 -norm and $\|D_+^s(u - u_N)\|$ (called “fractional norm” for simplicity, to be in accordance with the analysis), which can be computed from the expansion coefficients.

5.3.1. Numerical examples for FIVPs. We first consider the FIVP (5.1) with $f(x) = 1 + x + \cos x$. Note that the explicit form of the exact solution is not available, so we compute a reference exact solution by using the scheme (5.2) with large N .

In view of the error estimate in Theorem 5.1, we know that the errors decay exponentially, if the source term f is smooth, despite that the unknown solution is singular at $x = 1$. Indeed, we observe from Fig. 5.1 (left) that all errors decay exponentially, which verify our theoretical results that the convergence rate is only determined by the smoothness of the source term f . Indeed, we also see that the errors in the fractional norm for different s are indistinguishable, which again show that the convergence behaviour solely relies on regularity of f .

Next, we consider (5.1) with $s \in (1, 2)$ and the smooth exact solution: $u(x) = (1 - x^3)(1 - e^{1-x})$, and find the source term $f(x)$ from (5.1). It is clear that $f(x)$ is singular at $x = 1$, so our error analysis in Theorem 5.1 predicts that the convergence rate will be algebraic. Like in Remark 4.3, we calculate from u that

$$f^{(m)} = D_+^{s+m} u = O((1-x)^{2-s-m}), \quad x \rightarrow 1.$$

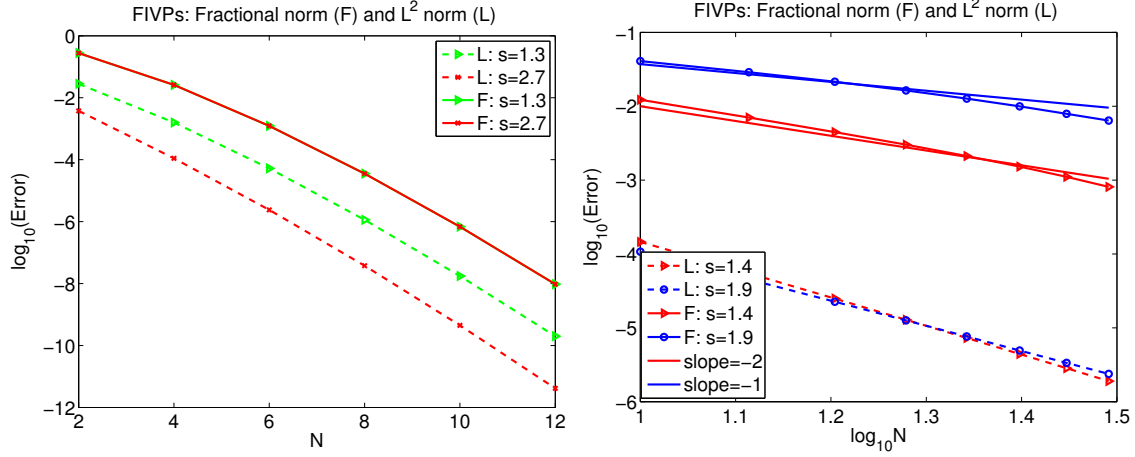


FIGURE 5.1. Convergence of the GJF-Petrov-Galerkin method for the FIVP (5.1). Left: (5.1) with smooth source term $f(x) = 1 + x + \cos x$. Right: (5.1) with smooth solution: $u(x) = (1 - x^3)(1 - e^{1-x})$.

Hence, in order to have $\|f^{(m)}\|_{\omega(m,m)} < +\infty$, we need

$$2(2 - s - m) + m > -1, \quad \text{i.e.,} \quad m < 5 - 2s, \quad m \in \mathbb{N}_0.$$

The convergence behaviours for different s are depicted in Fig. 5.1 (right). We see that the slopes of the lines agree very well with the theoretical estimates.

5.3.2. Numerical examples for FBVPs with integral boundary conditions. Now, we consider the FBVP (5.8) and its GJF-Petrov-Galerkin approximation (5.13). We first take $f(x) = \sin x$ in (5.8), and compute the reference exact solution as the previous case. We plot the errors for different

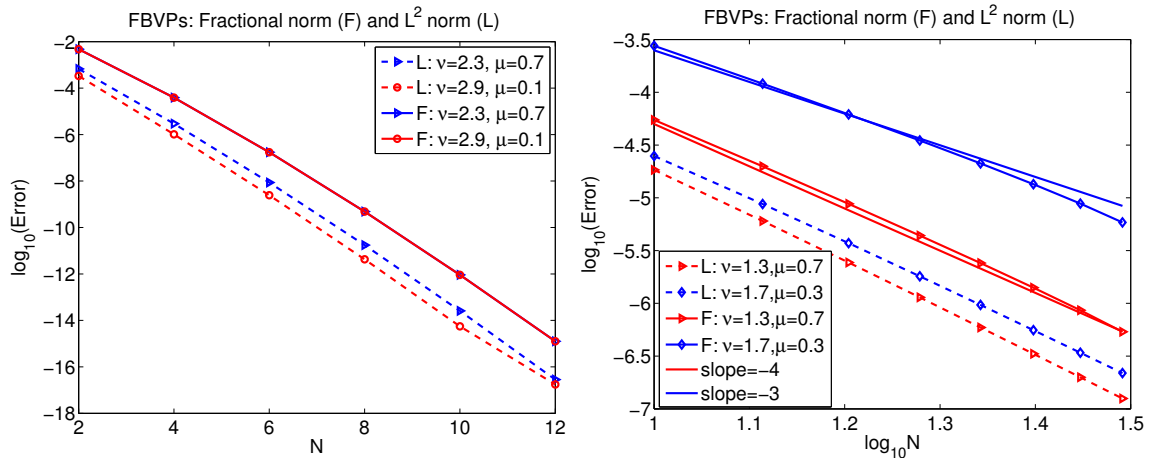


FIGURE 5.2. Convergence of the GJF-Petrov-Galerkin method for the FBVP (5.8). Left: $f(x) = \sin x$. Right: $u(x) = (1 - x)^2(1 - x - 6/(3 + \mu))$.

orders in Fig. 5.2 (left). As expected, the method is truly spectrally convergent, in agreement with the error estimate (5.20). Once again, the convergence rate only depends on the smoothness of f .

Next, we take the exact solution to be $u(x) = (1-x)^2(1-x-6/(3+\mu))$, and compute f from (5.8). We know from Theorem 5.2 that for $\nu \in (1, 2)$ with $\mu = 2 - \nu$, we have

$$\|D_+^{1-\mu}(u - u_N)\| \leq cN^{1-m}\|D_+^{m-\mu}u\|_{\omega^{(m-1, m-1)}}. \quad (5.56)$$

A direct calculation shows that, in order for $\|D_+^{m-\mu}u\|_{\omega^{(m-1, m-1)}} < \infty$, we require

$$2(2 - (m - \mu)) + m - 1 > -1, \quad \text{i.e., } m < 4 + 2\mu, \quad m \in \mathbb{N}.$$

Therefore, for $\nu = 1.3$, $\mu = 0.7$ and $\nu = 1.7$, $\mu = 0.3$, we have $m < 5.4$ and $m < 4.6$, respectively, and the expected convergence rate is $m - 1$. The numerical errors for this example are plotted in Fig. 5.2 (right). We observe that the convergence rates are consistent with our error estimates.

5.3.3. Numerical examples for FBVPs with homogeneous boundary conditions. As the last example, we consider the FBVP with homogeneous boundary conditions in (5.31). Similar to the previous cases, we first take a smooth source term $f(x) = xe^x$, and plot the errors in Fig. 5.3 (left), which shows an exponential convergence, as expected from the error estimates in Theorem 5.4.

Next, we take the exact solution $u = (1-x)\sin(\pi x)$ and compute f accordingly from (5.31). As before, we can derive from the error estimate (5.51) that the order of convergence m must satisfy $m < 5 - 2s$ with $m \in \mathbb{N}$. In Fig. 5.3 (right), we plot the errors for $\nu = 1.4$, $s = 0.4$ and $\nu = 1.9$, $s = 0.9$, respectively. We again see that the observed convergence rate agrees with the expected rate.

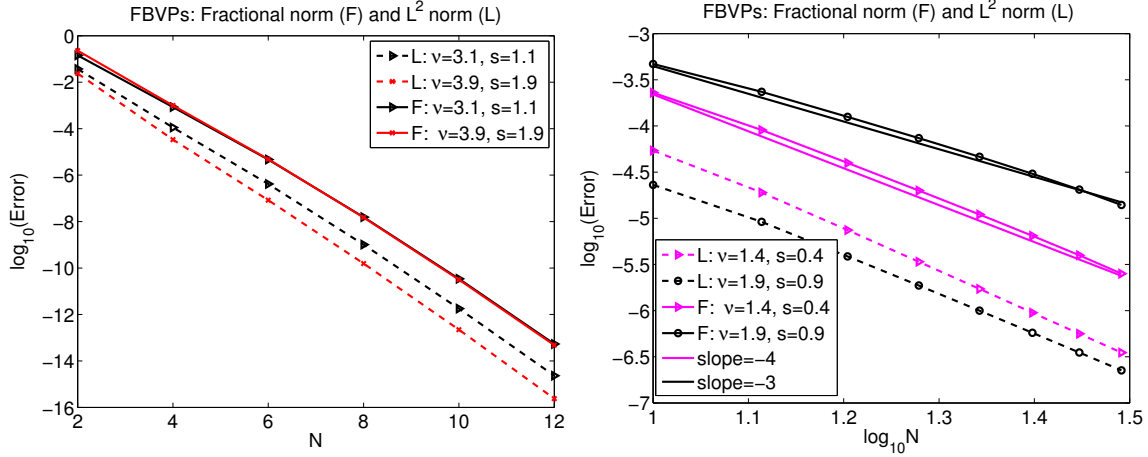


FIGURE 5.3. Convergence of the GJF-Petrov-Galerkin method for the FBVP (5.31). Left: $f(x) = xe^x$. Right: $u(x) = (1-x)\sin(\pi x)$.

Note that in all above examples, the L^2 errors are significant smaller than the errors in fractional norms. However, we cannot justify this rigorously. Unlike in the case of integer DEs where one can derive an improved error estimate in the L^2 -norm using a duality argument, we are unable to do this in the fractional case largely due to the lack of regularity in the usual Sobolev norm. Nevertheless, we see the gain of order in L^2 -norm from Theorem 4.1 in the context of approximation by GJFs.

6. EXTENSIONS, DISCUSSIONS AND CONCLUDING REMARKS

To conclude the paper, we show that the important formulas of Riemann-Liouville fractional derivatives can be extended in parallel to Caputo derivatives. Consequently, the analysis and results can be generalised to Caputo cases, and the GJFs enjoy similar remarkable approximability to Caputo FDEs. We also provide a summary of main contributions of the paper in the end of this section.

6.1. Extension to Caputo derivatives. It is seen that the formulas in Lemma 2.5 and Theorem 3.1 are exceedingly important in the preceding analysis and spectral algorithms involving Riemann-Liouville derivatives. Remarkably, similar results are also available for the Caputo derivatives.

Like Lemma 2.5, we have the following formulas involving Caputo derivatives.

Lemma 6.1. *Let $s \in [k-1, k)$ with $k \in \mathbb{N}$ and $x \in \Lambda$.*

- *For $\alpha > -1$ and $\beta \in \mathbb{R}$,*

$${}^C D_+^s \{ (1-x)^{\alpha+k} P_n^{(\alpha+k, \beta-k)}(x) \} = \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+k-s+1)} (1-x)^{\alpha+k-s} P_n^{(\alpha+k-s, \beta-k+s)}(x). \quad (6.1)$$

- *For $\alpha \in \mathbb{R}$ and $\beta > -1$,*

$${}^C D_-^s \{ (1+x)^{\beta+k} P_n^{(\alpha-k, \beta+k)}(x) \} = \frac{\Gamma(n+\beta+k+1)}{\Gamma(n+\beta+k-s+1)} (1+x)^{\beta+k-s} P_n^{(\alpha-k+s, \beta+k-s)}(x). \quad (6.2)$$

Proof. Let us first derive (6.1). In view of $D_+^k = (-1)^k D^k$ (cf. (2.7)), we obtain from (2.31) that

$$D^k \{ (1-x)^{\alpha+k} P_n^{(\alpha+k, \beta-k)}(x) \} = (-1)^k \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+1)} (1-x)^\alpha P_n^{(\alpha, \beta)}(x). \quad (6.3)$$

By Definition 2.1, we have ${}^C D_+^s v = (-1)^k I_+^{k-s} (D^k v)$, so using (2.29) with $\rho = k-s$ and (6.3) leads to

$$\begin{aligned} {}^C D_+^s \{ (1-x)^{\alpha+k} P_n^{(\alpha+k, \beta-k)}(x) \} &= \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+1)} I_+^{k-s} \{ (1-x)^\alpha P_n^{(\alpha, \beta)}(x) \} \\ &= \frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+k-s+1)} (1-x)^{\alpha+k-s} P_n^{(\alpha+k-s, \beta-k+s)}(x). \end{aligned}$$

This yields (6.1). The formula (6.2) can be derived similarly. \square

The counterpart of Theorem 3.1 takes a slightly different form in the range of parameters.

Theorem 6.1. *Let $s \in [k-1, k)$ with $k \in \mathbb{N}$ and $x \in \Lambda$.*

- *For $\alpha > k-1$ and $\beta \in \mathbb{R}$,*

$${}^C D_+^s \{ {}^+ J_n^{(-\alpha, \beta)}(x) \} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-s+1)} {}^+ J_n^{(-\alpha+s, \beta+s)}(x). \quad (6.4)$$

- *For $\alpha \in \mathbb{R}$ and $\beta > k-1$,*

$${}^C D_-^s \{ {}^- J_n^{(\alpha, -\beta)}(x) \} = \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta-s+1)} {}^- J_n^{(\alpha+s, -\beta+s)}(x). \quad (6.5)$$

Proof. With $(\alpha-k, \beta+k)$ in place of (α, β) in (6.1), we obtain (6.4) immediately from the definition (3.1). The rule (6.5) can be obtained in the same fashion. \square

Taking $s = \alpha$ in (6.4) leads to that for $\alpha > 0$ and $\beta \in \mathbb{R}$,

$${}^C D_+^s \{ {}^+ J_n^{(-\alpha, \beta)}(x) \} = \frac{\Gamma(n+\alpha+1)}{n!} P_n^{(0, \alpha+\beta)}(x). \quad (6.6)$$

Similarly, we derive from (6.5) an important formula, that is, for $\alpha \in \mathbb{R}$ and real $\beta > 0$,

$$D_-^\beta \{ {}^- J_n^{(\alpha, -\beta)}(x) \} = \frac{\Gamma(n+\beta+1)}{n!} P_n^{(\alpha+\beta, 0)}(x). \quad (6.7)$$

Indeed, the GJFs with parameter $\alpha > 0$ or $\beta > 0$ meets the conditions in (2.11), so we have the same formulas as in (3.14)-(3.15) for the Riemann-Liouville derivatives.

With the aid of the above derivative formulas, we can establish the GJF approximations in weighted Sobolev spaces, and develop efficient spectral methods for FDEs involving Caputo fractional derivatives accordingly. Here, we omit the details.

6.2. Discussions and concluding remarks. We considered in this paper spectral approximation of FDEs by introducing a class of priorly defined GJFs.

Our main contributions are twofold:

- Introduced a new class of GJFs, which extend the range of definition of polyfractomials [30] so that high-order fractional derivatives can be treated, revealed their relations with fractional derivatives, and studied their approximation properties.
- Constructed Petrov-Galerkin spectral methods for a class of prototypical FDEs, including arbitrarily high-order FIVPs and FBVPs which have not been numerically studied before, which led to sparse matrices, and derived error estimates with convergence rate only depending on the smoothness of data. In particular, if the data function is analytic, we obtain exponential convergence, despite the fact that the solution is singular.

The results presented in this paper indicate that, at least for the simple FDEs considered here, one can develop spectral methods to solve them with the same kind of computational complexity and accuracy as one solve for usual PDEs.

This is first but important step towards developing efficient and accurate spectral methods for solving FDEs. While we have only considered a class of very simple prototypical FDEs, the general principles and the approximation results developed in this paper open up new possibilities for dealing with more general FDEs.

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